講義録：領域とスペクトル

神保 秀一

北海道大学大学院　理学研究院　数学部門

非線形現象の数値シミュレーションと解析 (**March 05/ 2024**)

偏微分方程式や解の性質の領域依存性の研究

スペクトル解析は多くの諸科学で行われている (科学的背景) :

スペクトルは波の種類を区別する概念

$\Longrightarrow$ 波の波数(ベクトル)の 2 乗は作用素の固有値に対応

物体や媒質の (特異的) 形状や材質に対応し, スペクトル特性の研究は重要なステップ

音波 (楽器の空洞の仕組み, 音色) $\Longrightarrow$ 波動方程式

媒質の熱伝導, 物質の特性(形状, 均質化, 不純物) $\Longrightarrow$ 熱伝導方程式

建造物や機械の振動特性 $\Longrightarrow$ 弾性体波動方程式

電磁波の反射や吸収(アンテナ特性)$\Longrightarrow$ マクスウェル方程式

工学関連： 楽器の設計, 非破壊検査(不純物, 欠損), ビルの免震効果, アンテナ形状, ..

いくつかの (有名) 数学研究

１番の代表例は固有値の漸近分布 (ワイルの公式). 分布関数 $N(\lambda)$ (λ 以下の固有値の個数)

$$N(\lambda) = \sum_{k\in\mathbb{N},\lambda_k\leqq\lambda} 1 = \frac{\mathrm{vol}(\Omega)}{2^n\pi^{n/2}\Gamma((n/2)+1)}\lambda^{n/2} + O(\lambda^{(n-1)/2}) \quad (\lambda \gg 0)$$

ミニマックス原理および比較原理の応用として示される.(Courant-Hilbert book)

私の個人的印象深い研究：アダマールの領域研究 (1908) (固有値, グリーン関数の摂動公式), クーラン・ヒルベルトの本(ワイルの公式, 領域の変形と固有値の収束), シッファー・スペンサーの研究, スワンソン ('66), Beale('75), ラウチテーラー ('75) の研究, 小澤真 ('81, '83), 小澤真, アダマールの変分公式の研究, 東大修士論文 ('79), 非線型楕円型方程式等の解の構造 (Yamabe 方程式,..), 分岐やパターン形成の研究 (Matano, Vegas, Hale-Vegas, Dancer,..)　などにも応用. Maz'ya-Nazarov-Plamenevskii book...

ラプラシアンの固有値問題と変分法

$\Omega \subset \mathbb{R}^n$ 有界領域

$$\Delta\Phi + \lambda\,\Phi = 0 \quad \text{in} \quad \Omega, \qquad \Phi = 0 \quad \text{on} \quad \partial\Omega$$

固有値 $\{\lambda_k\}_{k=1}^{\infty}$, 固有関数$\{\Phi_k\}_{k=1}^{\infty}$ はどう決まるか？

[Rayleigh Quotient (レイリー商)]

$$\mathcal{R}(\Phi) = \int_\Omega |\nabla\Phi|^2 dx \Big/ \int_\Omega |\Phi|^2 dx \quad (\Phi \in X = H_0^1(\Omega))$$

$$\lambda_1 = \inf\{\mathcal{R}(\phi) \mid \phi \in X,\, \phi \not\equiv 0\}$$

最小化列 $\{\phi_p\}_{p=1}^{\infty} \subset H_0^1(\Omega)$ を任意に取る. このとき

$$\mathcal{R}(\phi_p) \downarrow \lambda_1 \geqq \tau \qquad (p \to \infty)$$

$$\exists c \geqq \int_\Omega |\nabla \phi_p|^2\, dx \geqq \lambda_1, \quad \int_\Omega |\phi_p|^2 dx = 1 \quad (p \geqq 1)$$

$$\int_\Omega |\nabla \phi_p|^2\, dx \leqq c + |\tau|$$

$\exists\ \{\phi_{p(j)}\}_{j=1}^{\infty} \subset \{\phi_p\}_{p=1}^{\infty} \subset H_0^1(\Omega), \exists \Phi_1 \in H_0^1(\Omega)$ s.t.

$$\phi_{p(j)} \rightharpoonup \Phi_1 \quad \text{in} \quad H_0^1(\Omega), \qquad \phi_{p(j)} \to \Phi_1 \quad \text{in} \quad L^2(\Omega), \quad \int_\Omega |\Phi_1|^2 dx = 1.$$

ノルムの弱下半連続性により

$$\lambda_1 = \liminf_{j\to\infty} \int_\Omega |\nabla \phi_{p(j)}|^2 dx \geqq \int_\Omega |\nabla \Phi_1|^2 dx.$$

$\mathcal{R}(\Phi_1) = \lambda_1$. Φ_1: 最小化元.

Lemma. Φ_1 は元の固有関数となり, λ_1 は固有値となる.

$$\int_\Omega (\nabla\Phi_1\nabla\varphi - \lambda_1\Phi_1\varphi)\,dx = 0 \quad (\varphi \in X = H_0^1(\Omega))$$

λ_1 を第 1 固有値, Φ_1 は対応する固有関数となる.

$$\lambda_2 = \inf\{\mathcal{R}(\phi) \mid \phi \in X,\ \phi \not\equiv 0,\, (\phi, \Phi_1)_{L^2(\Omega)} = 0\}$$

Φ_1 の直交補空間において同様の議論を繰り返すことにより, 第 2 固有値 $\lambda_2 \geqq \lambda_1$, と対応する固有関数 Φ_2 を得る. ここで $\Phi_2 \perp \Phi_1$, $\|\Phi_2\|_{L^2(\Omega)} = 1$ に注意する. この過程を繰り返すことで, 固有値列

$$\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots \leqq \lambda_m \leqq \cdots$$

と固有関数系を得る.

$$\Phi_1, \Phi_2, \Phi_3, \cdots, \Phi_m, \cdots$$

Proposition. 以上の固有値と固有関数に関し $\lim_{m\to\infty} \lambda_m = \infty$ が成立し, 固有関数系 $\{\Phi_m\}_{m=1}^{\infty}$ は $L^2(\Omega)$ で完全正規直交系となる.

ロバン条件あるいはノイマン条件の場合は前節の議論は(少し修正されるが), ほぼ同じ結果が成立する

$\Omega \subset \mathbb{R}^n$ は有界領域とする. 固有値問題を考える.

$$\Delta\Phi + \lambda\,\Phi = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial\Phi}{\partial\nu} + a(x)\Phi = 0 \quad \text{on} \quad \partial\Omega$$

$a(x) \geqq 0$ in $\partial\Omega$.

固有値および固有関数 $\{\lambda_k\}_{k=1}^{\infty}$, $\{\Phi_k\}_{k=1}^{\infty}$ は以下のレイリー商により前節と同じ議論で導かれる.

[Rayleigh Quotient]

$$\mathcal{R}(\Phi) = \left(\int_\Omega |\nabla\Phi|^2 dx + \int_{\partial\Omega} a(x)\Phi(x)^2 dS \right) \Big/ \int_\Omega |\Phi|^2 dx \quad (\Phi \in X = H^1(\Omega))$$

Max-min principle

ディリクレ境界条件, ロバン境界条件, ノイマン境界条件の共通して成立する, 固有値の特徴付け定理である最大最小原理を紹介しよう.

記号. $\mathcal{H}_i(\Omega)$ は $L^2(\Omega)$ の部分空間でその次元が高々 i 以下のもの全体とする. このとき固有値は以下のように表せる.

Theorem. $k \in \mathbb{N}$.

$$\lambda_k = \sup_{Y \in \mathcal{H}_{k-1}(\Omega)} \Big(\inf\{\mathcal{R}(\phi) \mid \phi \in X,\ \phi \not\equiv 0,\ \phi \perp Y \text{ in } L^2(\Omega)\} \Big)$$

(proof) 右辺の値を $\widetilde{\lambda}_k$ と書く.

$$\{\lambda_k\}_{k=1}^{\infty}, \quad \{\Phi_k\}_{k=1}^{\infty} \quad \text{previous argument}$$

(Step 1) まず $\lambda_k \leqq \widetilde{\lambda}_k$ を示す. 次の部分空間

$$Y = L.H.[\Phi_1, \cdots, \Phi_{k-1}]$$

は $k-1$ 次元である. よって $Y \in \mathcal{H}_{k-1}(\Omega)$.

$$\widetilde{\lambda}_k \geqq \inf\{\mathcal{R}(\phi) \mid \phi \in X,\ \phi \not\equiv 0,\ \phi \in Y^{\perp}\}$$

$$= \inf\{\mathcal{R}(\phi) \mid \phi \in X, \phi \not\equiv 0, (\phi, \Phi_1)_{L^2(\Omega)} = \cdots = (\phi, \Phi_{k-1})_{L^2(\Omega)} = 0\}$$

右辺は λ_k に一致する. そして $\widetilde{\lambda}_k \geqq \lambda_k$ が従う.
(Step 2) 次に部分空間

$$Y' = L.H.[\Phi_1, \cdots, \Phi_k]$$

とおく. 任意に $Y \in \mathcal{H}_{k-1}(\Omega)$ を取る. ここで次元定理などにより $Y' \cap Y^{\perp} \neq \{\mathbf{0}\}$ となるから

$$\exists v(x) = \sum_{j=1}^{k} c_j \Phi_j(x) \in Y' \cap Y^{\perp}, \quad (c_1, c_2, \cdots, c_k) \neq (0, 0, \cdots, 0)$$

が存在する. $\inf\{\mathcal{R}(\phi) \mid \phi \in X,\, \phi \not\equiv 0,\, \phi \in Y^{\perp}\} \leqq \mathcal{R}(v)$ となるので

$$\mathcal{R}(v) = \left(\sum_{j=1}^{k} \lambda_j |c_j|^2 \right) \Bigg/ \left(\sum_{j=1}^{k} |c_j|^2 \right) \leqq \lambda_k$$

が従う. よって

$$\inf\{\mathcal{R}(\phi) \mid \phi \in X,\, \phi \not\equiv 0,\, \phi \in Y^{\perp}\} \leqq \lambda_k$$

$Y \in \mathcal{H}_{k-1}(\Omega)$ で上限を取って, 次を得る.

$$\widetilde{\lambda}_k = \sup_{Y \in \mathcal{H}_{k-1}(\Omega)} \big(\inf\{\mathcal{R}(\phi) \mid \phi \in X,\, \phi \not\equiv 0,\, \phi \in Y^{\perp}\}\big) \leqq \lambda_k.$$

□

[Dirichlet 固有値の領域単調性]

ディリクレ境界条件の場合には領域の包含に関する大小が固有値の小大を導くという著しい性質を述べる.

Theorem. $\Omega_1 \subset \Omega_2$ を仮定する. $\{\lambda_{j,k}\}_{k=1}^{\infty}$ を Ω_j $(j=1,2)$ に対する固有値とする. このとき $\forall k \geqq 1$ に対して, $\lambda_{2,k} \leqq \lambda_{1,k}$ が成立する.

(Proof) 記号を導入する. $\psi \in L^2(\Omega_1)$ に対し,

$$\widetilde{\psi}(x) = \begin{cases} \psi(x) & (x \in \Omega_1) \\ 0 & (x \in \Omega_2 \setminus \Omega_1) \end{cases}$$

とする. $Z \subset L^2(\Omega_2)$ に対し, 関連する $L^2(\Omega_1)$ の部分空間を

$$Z_{|\Omega_1} = \{\Psi_{|\Omega_1} \in L^2(\Omega_1) \mid \Psi \in Z\}$$

で定める. もし $p \geqq 0$ ならば写像

$$\mathcal{H}_p(\Omega_2) \ni Z \longmapsto Z_{|\Omega_1} \in \mathcal{H}_p(\Omega_1)$$

は全射である.

これによって比較による次の不等式を得る.

$$\begin{aligned}\lambda_k(\Omega_2) &= \sup_{Z\in\mathcal{H}_{k-1}(\Omega_2)} \Big(\inf\{\mathcal{R}_2(\Phi) \mid \Phi \in H_0^1(\Omega_2),\ \Phi \not\equiv 0,\ \Phi \perp_2 Z\}\Big) \\ &\leqq \sup_{Z\in\mathcal{H}_{k-1}(\Omega_2)} \Big(\inf\{\mathcal{R}_2(\widetilde{\phi}) \mid \phi \in H_0^1(\Omega_1),\ \phi \not\equiv 0,\ \widetilde{\phi} \perp_2 Z\}\Big) \\ &= \sup_{Z\in\mathcal{H}_{k-1}(\Omega_2)} \Big(\inf\{\mathcal{R}_1(\phi) \mid \phi \in H_0^1(\Omega_1),\ \phi \not\equiv 0,\ \phi \perp_1 Z_{|\Omega_1}\}\Big) \\ &= \sup_{Y\in\mathcal{H}_{k-1}(\Omega_1)} \Big(\inf\{\mathcal{R}_1(\phi) \mid \phi \in H_0^1(\Omega_1),\ \phi \not\equiv 0,\ \phi \perp_1 Y\}\Big) = \lambda_k(\Omega_1)\end{aligned}$$

以上は [4] 参照で詳述されている.

特異的な領域変形と楕円型作用素(ラプラシアン)

本節ではパラメータに依存する領域が, なんらかの特異性をもつ変形をして同相でない集合に収束するケースを扱う. まず有界領域 $\Omega \subset \mathbb{R}^n$ を固定する.

摂動領域 $\Omega(\epsilon)$ $(\epsilon > 0)$ の 2 つのタイプ

(A) $\Omega(\epsilon)$: 小さな穴あるいは欠損(トンネル)あり (with some B.C. for emerging boundary).
$\Omega(\epsilon)$: $\epsilon \to 0$ の過程で増加

(B) $\Omega(\epsilon)$ の一部が低次元集合に退化 (Neumann B.C.)
$\Omega(\epsilon)$: $\epsilon \to 0$ の過程で減少

固有値 $\lambda_k(\epsilon)$ の挙動を調べたい.

References:

(A) : Swanson ('63,'77), Rauch-Taylor('75), Chavel-Feldman ('78), Ozawa ('81, '83), V.Maz'ya-S.Nazarov-B.Plamenevskij('85), Flucher('85), Berard..., Besson('85), Chavel-Feldman('88), ..., J.('15),..

(B) : Beale ('75), Chavel-Feldman(78', '81), Berard-Gallot('83), .., J. ('93), Gadylshin('93), Arrieta ('94), ..., Gadylshin('05), J.-Kosugi ('09), ...

Related with this talk:

[0] Swanson, Asymptotic variational formulae for eigenvalues, Canad. Math. Bull. 6 (1963).

[1] S. Jimbo, Eigenvalues of the Laplacian in a domain with a thin tubular hole, J. Elliptic, Parabolic, Equations **1** (2015).

[2] S. Jimbo and S. Kosugi, Spectra of domains with partial degeneration, J. Math. Sci. Univ. Tokyo, **16** (2009).

[3] V.Maz'ya, S.Nazarov, B.Plamenevskij, Asymptotic theory of elliptic boundary value problems in singularly perturbed domains, I, II, Birkhäuser 2000.

[4] 神保, 偏微分方程式入門, 共立出版, 2006.

小さな穴をもつ領域のDirichlet固有値の挙動

$\Omega \subset \mathbb{R}^n$ $(n \geqq 2)$ 有界領域. $O \in \Omega$ 固定点. Ω, $\Omega(\epsilon) = \Omega \setminus B(O,\epsilon)$ の第 k 番目のDirichlet 固有値 $\lambda_k(\epsilon), \lambda_k$ の差は小さいことを見る. $\lambda_k \leqq \lambda_k(\epsilon)$ はすでに判明している.

Proposition. $\lim_{\epsilon\to 0} \lambda_k(\epsilon) = \lambda_k$.
(proof)

$\{\lambda_k\}_{k=1}^\infty$: Ωにおけるディリクレ固有値 $\{\Phi_k\}_{k=1}^\infty$: 対応する固有関数の正規直交系 $L^2(\Omega)$

$$\{\lambda_k(\epsilon)\}_{k=1}^\infty : \ \Omega(\epsilon) \text{ ディリクレ境界条件下の固有値}$$

カットオフ関数: $r_0 > 0$ を小さく取り $\overline{B(O,r_0)} \subset \Omega$ となるようにしておく. 以下の関数を用意する.

$$\rho_\epsilon(x) = \frac{\log(|x|/\epsilon)}{\log(r_0/\epsilon)} \quad \text{for} \quad x \in \mathbb{R}^n, \epsilon \leqq |x| \leqq r_0.$$

単純な計算で以下を得る.

$$\int_{\epsilon \leqq |x| \leqq r_0} |\rho_\epsilon(x) - 1|^2 dx = O(\frac{1}{(\log\epsilon)^2}), \quad \int_{\epsilon \leqq |x| \leqq r_0} |\nabla \rho_\epsilon(x)|^2 dx = \begin{cases} O(\frac{1}{|\log\epsilon|}) & (n=2) \\ O(\frac{1}{(\log\epsilon)^2}) & (n \geqq 3) \end{cases}$$

$(0 < \epsilon \leqq r_0)$. ここで

$$\widetilde{\phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) \quad \text{for} \quad x \in \Omega \setminus B(O, r_0) \\ \rho_\epsilon(x)\Phi_k(x) \quad \text{for} \quad x \in B(O, r_0) \setminus B(O, \epsilon) \end{cases}$$

$\widetilde{\phi}_{k,\epsilon} \in H^1(\Omega(\epsilon))$, $\widetilde{\phi}_{k,\epsilon} = 0$ on $\Omega(\epsilon)$. 再び直接計算で次を得る.

$$(\widetilde{\phi}_{p,\epsilon}, \widetilde{\phi}_{\ell,\epsilon})_{L^2(\Omega(\epsilon))} = \delta(p,\ell) + \kappa'(p,\ell,\epsilon), \quad (\nabla\widetilde{\phi}_{p,\epsilon}, \nabla\widetilde{\phi}_{\ell,\epsilon})_{L^2(\Omega(\epsilon))} = \lambda_p\delta(p,\ell) + \kappa(p,\ell,\epsilon).$$

ただし

$$\kappa'(p,\ell,\epsilon) = O(\frac{1}{|\log\epsilon|}), \quad \kappa(p,\ell,\epsilon) = O(\frac{1}{|\log\epsilon|^{1/2}}) \quad \text{for} \quad 1 \leqq p, \ell \leqq k$$

である. 最大最小原理により $\lambda_k(\epsilon)$ を評価する. まず

$$\widetilde{E}_\epsilon = \text{L.H.}[\widetilde{\phi}_{1,\epsilon}, \widetilde{\phi}_{2,\epsilon}, \cdots, \widetilde{\phi}_{k,\epsilon}]$$

とすると $\dim\widetilde{E}_\epsilon = k$ for small $\epsilon > 0$ なぜなら

$$(\widetilde{\phi}_{p,\epsilon}, \widetilde{\phi}_{\ell,\epsilon})_{L^2(\Omega(\epsilon))} = \delta(p,\ell) + O(|\log\epsilon|^{-1})$$

for $p, \ell \geqq 1$ であるから．さて任意に部分空間 $E \subset L^2(\Omega(\epsilon))$ を $\dim E \leqq k-1$ となるように取る．そうすると非自明な関数

$$\varphi(x) = \sum_{\ell=1}^{k} c_\ell \, \widetilde{\phi}_{\ell,\epsilon}(x) \in \widetilde{E}_\epsilon$$

s.t. $\varphi \perp E$ in $L^2(\Omega(\epsilon))$ が存在する．よって

$$\inf\{\mathcal{R}_\epsilon(\Phi) \mid \Phi \in H^1(\Omega(\epsilon)), \Phi_{|\Gamma} \equiv 0, \Phi \perp E \,\mathrm{in}\, L^2(\Omega(\epsilon))\}$$

$$\leqq R_\epsilon(\varphi) = \int_{\Omega(\epsilon)} |\nabla\varphi|^2 dx / \|\varphi\|^2_{L^2(\Omega(\epsilon))}.$$

斉次性により一般性を失うことなく $\sum_{\ell=1}^{k} |c_k|^2 = 1$ として良い (i.e. $\mathcal{R}_\epsilon(t\phi) = \mathcal{R}_\epsilon(\phi)$ $(t > 0)$)．ここで

$$(\nabla\Phi_{\ell_1}, \nabla\Phi_{\ell_2})_{L^2(\Omega)} = \lambda_{\ell_1}\delta(\ell_1, \ell_2), \quad (\Phi_{\ell_1}, \Phi_{\ell_2})_{L^2(\Omega))} = \delta(\ell_1, \ell_2)$$

$$\int_{\Omega(\epsilon)} |\nabla\varphi|^2 dx = \sum_{\ell_1,\ell_2=1}^{k} \int_{\Omega(\epsilon)} c_{\ell_1} c_{\ell_2} \nabla\Phi_{\ell_1} \nabla\Phi_{\ell_2} dx = \sum_{\ell_1,\ell_2=1}^{k} (\lambda_{\ell_1}\delta(\ell_1, \ell_2) + \kappa(\ell_1, \ell_2, \epsilon)) c_{\ell_1} c_{\ell_2}$$

$$= \sum_{\ell=1}^{k} \lambda_\ell c_\ell^2 + \sum_{\ell_1,\ell_2=1}^{k} \kappa(\ell_1,\ell_2,\epsilon) c_{\ell_1} c_{\ell_2} \leqq \lambda_k \sum_{\ell=1}^{k} c_\ell^2 + \sum_{\ell_1,\ell_2=1}^{k} \kappa(\ell_1,\ell_2,\epsilon) c_{\ell_1} c_{\ell_2}$$

$$\int_{\Omega(\epsilon)} |\varphi|^2 dx = \sum_{\ell_1,\ell_2=1}^{k} \int_{\Omega(\epsilon)} c_{\ell_1} c_{\ell_2} \Phi_{\ell_1} \Phi_{\ell_2} dx = \sum_{\ell_1,\ell_2=1}^{k} (\delta(\ell_1,\ell_2) + \kappa'(\ell_1,\ell_2,\epsilon)) c_{\ell_1} c_{\ell_2}$$

$$= \sum_{\ell=1}^{k} c_\ell^2 + \sum_{\ell_1,\ell_2=1}^{k} \kappa'(\ell_1,\ell_2,\epsilon) c_{\ell_1} c_{\ell_2}$$

Since $\sum_{\ell=1}^{k} c_\ell^2 = 1$, これにより

$$\mathcal{R}_\epsilon(\varphi) \leqq \frac{\lambda_k + \sum_{\ell_1,\ell_2=1}^{k} |\kappa(\ell_1,\ell_2,\epsilon)|}{1 - \sum_{\ell_1,\ell_2=1}^{k} |\kappa'(\ell_1,\ell_2,\epsilon)|}$$

右辺は E の取り方によらないので上限を取って, 次の評価を得る.

(1)
$$\lambda_k(\epsilon) \leqq \frac{\lambda_k + \sum_{\ell_1,\ell_2=1}^{k} |\kappa(\ell_1,\ell_2,\epsilon)|}{1 - \sum_{\ell_1,\ell_2=1}^{k} |\kappa'(\ell_1,\ell_2,\epsilon)|} = \lambda_k + O(|\log\epsilon|^{-1/2})$$

結局 $\lambda_k \leqq \lambda_k(\epsilon) \leqq \lambda_k + o(1)$ が成立する. □

(A1) 小さな穴をもつ領域, 固有値の挙動の精密化

$\Omega \subset \mathbb{R}^n$ を有界領域として, 次の固有値問題を扱う.

$$(1) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega, \qquad \Phi = 0 \quad \text{on} \quad \Gamma = \partial\Omega.$$

$\{\lambda_k\}_{k=1}^{\infty}$: 固有値 (多重度を数え単調非減少に並べられている)

$\boldsymbol{a} \in \Omega$ を任意の点として固定する.

$$\Omega(\epsilon) = \Omega \setminus \overline{B(\boldsymbol{a}, \epsilon)}, \quad \Gamma(\epsilon) = \partial B(\boldsymbol{a}, \epsilon), \quad \Gamma = \partial\Omega.$$

Dirichlet B.C. on $\Gamma(\epsilon)$

$$(2-D) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \text{ on } \Gamma(\epsilon) \cup \Gamma$$

Neumann B.C. on $\Gamma(\epsilon)$

$$(2-N) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \text{ on } \Gamma, \quad \partial\Phi/\partial\nu = 0 \text{ on } \Gamma(\epsilon)$$

固有値問題 (2-D), (2-N) の第 k 固有値をそれぞれ $\lambda_k^D(\epsilon)$, $\lambda_k^N(\epsilon)$ と書く.

Shin Ozawa ('81,'83) の仕事.

Theorem ($n=2$ or $n=3$). (1) の固有値 λ_k は単純であるとする. このとき, 次が成立する.

$$\lambda_k^D(\epsilon)-\lambda_k=\begin{cases}4\pi\Phi_k(\boldsymbol{a})^2\epsilon+\text{H.O.T.} & (n=3)\\ (2\pi/\log(1/\epsilon))\Phi_k(\boldsymbol{a})^2+\text{H.O.T.} & (n=2)\end{cases}$$

Theorem ($n=2$ or $n=3$). (1) の固有値 λ_k は単純であると仮定する. このとき, 次が成立する.

$$\lambda_k^N(\epsilon)-\lambda_k=\begin{cases}\pi(-2|\nabla\Phi_k(\boldsymbol{a})|^2+(4\lambda_k/3)\Phi_k(\boldsymbol{a})^2)\epsilon^3+\text{H.O.T.} & (n=3)\\ \pi\left(-2|\nabla\Phi_k(\boldsymbol{a})|^2+\lambda_k\Phi_k(\boldsymbol{a})^2\right)\epsilon^2+\text{H.O.T.} & (n=2)\end{cases}$$

小澤真の方法 : グリーン関数のシッファースペンサー型の摂動公式を応用した近似グリーン作用素を構成してグリーン作用素の固有値の摂動公式を求める流儀. 今の時代から見れば次元の制約が生じることなどが弱点に見える. 1980年頃の時代で良い方法が見えない時代に困難な課題を解決した点は画期的であると思われる. (cf. Ozawa ('83,'92), Roppongi ('93), Ozawa-Roppongi ('92)).

Swanson method : Swanson ('63) の仕事はDirichlet条件の場合に小澤氏の仕事より先んじて一般的な良い方法である. ただし, 摂動公式までには到達していない.

$n=2$ で**Neumann B.C.** の場合で証明法を概説する.

$\Omega\subset\mathbb{R}^2$ 有界領域とし $\boldsymbol{a}\in\Omega$ を固定点とする.

(1) の固有値, 固有関数が

$$\{\lambda_k\}_{k=1}^{\infty} \quad , \quad \{\Phi_k\}_{k=1}^{\infty} \quad \text{with} \quad (\Phi_p,\Phi_q)_{L^2(\Omega)}=\delta(p,q)$$

を満たすとする.

(2-N) の固有値,固有関数が

$$\{\lambda_k^N(\epsilon)\}_{k=1}^{\infty}, \quad \{\Phi_{k,\epsilon}\}_{k=1}^{\infty} \quad \text{with} \quad (\Phi_{p,\epsilon},\Phi_{q,\epsilon})_{L^2(\Omega(\epsilon))}=\delta(p,q)$$

を満たすとする.

$\lambda_k(\epsilon)-\lambda_k$ を精密に見たい. そのためするべきことは以下の 2 つのことである.

(i) 真の固有関数 $\Phi_{k,\epsilon}$ の特徴付け

(ii) 良い近似固有関数 $\widetilde{\Phi}_{k,\epsilon}$ の構成

(i) $\Phi_{k,\epsilon}$ の特徴付け

Proposition. 任意の正数列 $\{\epsilon(p)\}_{p=1}^{\infty}$ で $\lim_{p\to\infty}\epsilon(p)=0$ となるものに対し，部分列 $\{\epsilon(p(q))\}_{q=1}^{\infty}$ と列 $\{\lambda'_k\}_{k=1}^{\infty}$ および完全正規直交系 $\{\Phi'_k\}_{k=1}^{\infty}\subset L^2(\Omega)$ があって

$$\Delta\Phi'_k+\lambda'_k\Phi'_k=0 \text{ in } \Omega, \quad \Phi'_k=0 \text{ on } \partial\Omega,$$

および

$$\lim_{q\to\infty}\lambda_k(\epsilon(p(q)))=\lambda'_k, \quad \lim_{q\to\infty}\sup_{x\in\Omega(\epsilon(p(q)))}|\Phi_{k,\epsilon(p(q))}(x)-\Phi'_k(x)|=0 \quad (\forall k\in\mathbb{N})$$

となる.

証明は概略のみ．穴から離れた領域での固有関数の評価と収束性の検証．バリア関数によって穴の近くでの評価をする．$\Omega(\epsilon)$ 全体での一様収束性の検証.

以上を用いて次の結果を得る.

Proposition. $\lambda_k=\lambda'_k$ $(k\geqq 1)$, $\lim_{\epsilon\to 0}\lambda_k(\epsilon)=\lambda_k$.

(ii) 近似固有関数 $\widetilde{\Phi}_{k,\epsilon}$ の構成. 固定領域 Ω における固有関数 Φ_k を補正する. これは穴 $B(\boldsymbol{a},\epsilon)$ の縁でディリクレ条件をみたすように巧みにカットオフすることを考える. まず次の関数を用意する.

$$\eta_k(x) = \frac{\langle \nabla\Phi_k(\boldsymbol{a}), x-\boldsymbol{a}\rangle}{|x-\boldsymbol{a}|^2} \quad \text{(harmonic in} \quad \mathbb{R}^2 \setminus \{\boldsymbol{a}\}).$$

まず直接計算で簡単にわかることから始める.

$$\nabla\eta_k(x) = \frac{\nabla\Phi_k(\boldsymbol{a})}{|x-\boldsymbol{a}|^2} + \langle \nabla\Phi_k(\boldsymbol{a}), x-\boldsymbol{a}\rangle \frac{(x-\boldsymbol{a})}{|x-\boldsymbol{a}|}\frac{(-2)}{|x-\boldsymbol{a}|^3}$$

$$= \frac{\nabla\Phi_k(\boldsymbol{a})}{|x-\boldsymbol{a}|^2} - 2\langle \nabla\Phi_k(\boldsymbol{a}), x-\boldsymbol{a}\rangle \frac{(x-\boldsymbol{a})}{|x-\boldsymbol{a}|^4}$$

$$\Delta\eta_k(x) = \frac{-2\langle \nabla\Phi_k(\boldsymbol{a}), x-\boldsymbol{a}\rangle}{|x-\boldsymbol{a}|^4} - 2\langle \nabla\Phi_k(\boldsymbol{a}), \frac{x-\boldsymbol{a}}{|x-\boldsymbol{a}|^4}\rangle$$

$$-2\langle \nabla\Phi_k(\boldsymbol{a}), x-\boldsymbol{a}\rangle \left(\frac{2}{|x-\boldsymbol{a}|^4} - \frac{4|x-\boldsymbol{a}|^2}{|x-\boldsymbol{a}|^6} \right) = 0 \qquad (x \neq \boldsymbol{a})$$

ここで $\widetilde{\Phi}_{k,\epsilon}$ を定める.

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) + \epsilon^2 \eta_k(x) & (x \in B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon)) \\ \Phi_k(x) + \epsilon^2 \widehat{\eta}_k(x) & (x \in \Omega \setminus B(\boldsymbol{a}, r_0)) \end{cases}$$

ただし $\widehat{\eta}_k$ 次のラプラス方程式の解として定める. すなわち $\widehat{\eta}$ は

$$\Delta \widehat{\eta} = 0 \text{ in } \Omega \setminus B(\boldsymbol{a}, r_0), \qquad \widehat{\eta}(x) = 0 \text{ on } \partial\Omega, \qquad \widehat{\eta}(x) = \eta_k(x) \text{ on } \partial B(\boldsymbol{a}, r_0).$$

とする.

さて摂動計算の本道に戻る. 固有値方程式の弱形式を用いる.

$$\int_{\Omega(\epsilon)} (\langle \nabla \Phi_{k,\epsilon}, \nabla \varphi \rangle - \lambda_k(\epsilon) \Phi_{k,\epsilon} \varphi) dx = 0 \qquad (\forall \varphi \in H^1(\Omega(\epsilon)) \text{ with } \varphi = 0 \text{ on } \partial\Omega)$$

近似固有関数 $\varphi = \widetilde{\Phi}_{k,\epsilon}$ により,

$$\int_{\Omega(\epsilon)} (\langle \nabla \Phi_{k,\epsilon}, \nabla \widetilde{\Phi}_{k,\epsilon} \rangle - \lambda_k(\epsilon) \Phi_{k,\epsilon} \widetilde{\Phi}_{k,\epsilon}) dx = 0$$

となる.

Swanson trick :

積分等式を詳細に見る. $\lambda_k(\epsilon) - \lambda_k$ を抽出する.

$$\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \langle \nabla \Phi_{k,\epsilon}, \nabla(\Phi_k + \epsilon^2 \eta_k) \rangle dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \langle \nabla \Phi_{k,\epsilon}, \nabla(\Phi_k + \epsilon^2 \widehat{\eta}_k) \rangle dx$$
$$-\lambda_k(\epsilon) \left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon} (\Phi_k + \epsilon^2 \eta_k) dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon} (\Phi_k + \epsilon^2 \widehat{\eta}_k) dx \right) = 0$$

Gauss-Green 公式より

$$\int_{\partial(B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon))} \Phi_{k,\epsilon} \frac{\partial}{\partial \nu}(\Phi_k + \epsilon^2 \eta_k) dS - \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon} \Delta \Phi_k dx$$
$$+ \int_{\partial(\Omega\setminus B(\boldsymbol{a},r_0))} \Phi_{k,\epsilon} \frac{\partial}{\partial \nu}(\Phi_k + \epsilon^2 \widehat{\eta}_k) dS - \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon} \Delta \Phi_k dx$$
$$-\lambda_k(\epsilon) \left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon} (\Phi_k + \epsilon^2 \eta_k) dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon} (\Phi_k + \epsilon^2 \widehat{\eta}_k) dx \right) = 0$$

$\Delta\Phi_k = -\lambda_k\Phi_k$ より

$$
\begin{aligned}
(\lambda_k(\epsilon) - \lambda_k)\int_{\Omega(\epsilon)} \Phi_{k,\epsilon}\Phi_k dx &= \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k + \epsilon^2\eta_k)dS \\
&+ \int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k + \epsilon^2\eta_k)dS + \int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu_2}(\Phi_k + \epsilon^2\widehat{\eta}_k)dS \\
&-\lambda_k(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\,(\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\,(\epsilon^2\widehat{\eta}_k)dx\right)
\end{aligned}
$$

$$=: I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon)$$

ν_1 is the unit outward normal vector of $\partial(B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon))$
ν_2 is the unit outward normal vector of $\partial(\Omega \setminus B(\boldsymbol{a}, r_0))$.

右辺の評価より次がわかる.

$$(\lambda_k(\epsilon)-\lambda_k)(\Phi_{k,\epsilon},\Phi_k)_{L^2(\Omega(\epsilon))}=O(\epsilon^2)$$

これから当然ながら以下の結果が従う.

$$\lim_{\epsilon\to 0}\lambda_k(\epsilon)=\lambda_k \quad (\forall k \geqq 1).$$

より精密な摂動公式を求めて議論が続く.

$$\frac{1}{\epsilon^2}I_1(\epsilon)=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k+\epsilon^2\eta_k)dS=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\langle\nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS$$

$$=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi'_k\langle\nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS+\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}(\Phi_{k,\epsilon}-\Phi'_k)\langle\nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS$$

$$=: \widetilde{I}_1(\epsilon)+o(1)$$

$\partial B(\boldsymbol{a},\epsilon)$ (i.e. $|x-\boldsymbol{a}|=\epsilon$) より

$$\frac{\partial}{\partial\nu_1}\widetilde{\Phi}_{k,\epsilon}=\frac{\partial}{\partial\nu_1}(\Phi_k+\epsilon^2\eta_k)=\langle\nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle=O(\epsilon)$$

$\epsilon = \epsilon(p(q))$ に対し,

$$I_3(\epsilon) = -\lambda_k \epsilon^2 \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_k' \eta_k dx - \lambda_k \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_k' \widehat{\eta}_k dx + o(\epsilon^2)$$

を得る.

$$= \epsilon^2 \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Delta\Phi_k' \eta_k dx + \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Delta\Phi_k' \widehat{\eta}_k dx + o(\epsilon^2)$$

$$\widetilde{I}_3(\epsilon) := \frac{1}{\epsilon^2} I_3(\epsilon) = \int_{\partial(B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon))} \left(\frac{\partial \Phi_k'}{\partial \nu_1}\eta_k - \Phi_k' \frac{\partial \eta_k}{\partial \nu_1}\right) dS + \int_{\partial(\Omega\setminus B(\boldsymbol{a},r_0))} \left(\frac{\partial \Phi_k'}{\partial \nu_2}\widehat{\eta}_k - \Phi_k' \frac{\partial \widehat{\eta}_k}{\partial \nu_2}\right) dS + o(1)$$

$$= \int_{\partial B(\boldsymbol{a},\epsilon)} \left(\frac{\partial \Phi_k'}{\partial \nu_1}\eta_k - \Phi_k' \frac{\partial \eta_k}{\partial \nu_1}\right) dS + \int_{\partial B(\boldsymbol{a},r_0)} \left(\frac{\partial \Phi_k'}{\partial \nu_1}\eta_k - \Phi_k' \frac{\partial \eta_k}{\partial \nu_1}\right) dS + \int_{\partial B(\boldsymbol{a},r_0)} \left(\frac{\partial \Phi_k'}{\partial \nu_2}\widehat{\eta}_k - \Phi_k' \frac{\partial \widehat{\eta}_k}{\partial \nu_2}\right) dS + o(1)$$

$$\begin{aligned}\widetilde{I}_2(\epsilon) := \frac{1}{\epsilon^2} I_2(\epsilon) &= \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial}{\partial \nu_1}(\Phi_k + \epsilon^2 \eta_k)dS + \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial}{\partial \nu_2}(\epsilon^2 \widehat{\eta}_k)dS \\ &= \int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial \eta_k}{\partial \nu_1}dS + \int_{\partial B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\frac{\partial \widehat{\eta}_k}{\partial \nu_2}dS \\ &= \int_{\partial B(\boldsymbol{a},r_0)} \Phi'_k\frac{\partial \eta_k}{\partial \nu_1}dS + \int_{\partial B(\boldsymbol{a},r_0)} \Phi'_k\frac{\partial \widehat{\eta}_k}{\partial \nu_2}dS + o(1)\end{aligned}$$

$\widetilde{I}_2(\epsilon)$, $\widetilde{I}_3(\epsilon)$ から,次を見る.

$$\widetilde{I}_2(\epsilon) + \widetilde{I}_3(\epsilon) = \int_{\partial B(\boldsymbol{a},\epsilon)} \frac{\partial \Phi'_k}{\partial \nu_1}\eta_k dS - \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi'_k\frac{\partial \eta_k}{\partial \nu_1}dS + o(1)$$

Lemma.

$$\widetilde{I}_1(\epsilon) = \lambda_k \pi \Phi'_k(\boldsymbol{a})\Phi_k(\boldsymbol{a}) + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0),$$

$$\widetilde{I}_2(\epsilon) + \widetilde{I}_3(\epsilon) = -2\pi\langle \nabla\Phi'_k(\boldsymbol{a}), \nabla\Phi_k(\boldsymbol{a})\rangle + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0),$$

(Sketch of the proof) テーラー展開を用いて得られる. □

結局次を得る.

$$\lim_{q\to\infty} \frac{\lambda_k(\epsilon(p(q))) - \lambda_k}{\epsilon(p(q))^2}(\Phi_{k,\epsilon(p(q))}, \Phi_k)_{L^2(\Omega(\epsilon(p(q))))}$$
$$= \pi(-2\langle \nabla\Phi'_k(\boldsymbol{a}), \nabla\Phi_k(\boldsymbol{a})\rangle + \lambda_k\pi\Phi'_k(\boldsymbol{a})\Phi_k(\boldsymbol{a}))$$

λ_k は単純より, $\Phi'_k = \Phi_k$ または $\Phi'_k = -\Phi_k$ であり, $\{\epsilon(p)\}_{p=1}^{\infty}$ 任意であったから

$$\lim_{\epsilon\to 0} \frac{\lambda_k(\epsilon) - \lambda_k}{\epsilon^2} = \pi(-2|\nabla\Phi_k(\boldsymbol{a})|^2 + \lambda_k\pi\Phi_k(\boldsymbol{a})^2)$$

を結論できる.

以下本稿は英語で記述する.

(A1) Generalization : Domain with a thin tubular hole

Hereafter we deal with more general case of singular deformation of domains which has holes of thin tubular shape. Let M be a $m-$dimensional smooth compact orientable manifold such that $M \subset \Omega$ and $0 \leqq m \leqq n-2$ and put

$$B(M,\epsilon) = \{x \in \mathbb{R}^n \mid \ \mathrm{dist}(x,M) < \epsilon\}, \qquad \Gamma = \partial\Omega, \quad \Gamma(M,\epsilon) = \partial B(M,\epsilon).$$

Note $|B(M,\epsilon)| = O(\epsilon^{n-m})$.

Let $\Omega(\epsilon) = \Omega \setminus \overline{B(M,\epsilon)}$ and $\lambda_k^D(\epsilon)$ be the $k-$th eigenvalue of the Laplacian in $\Omega(\epsilon)$ with the Dirichlet B.C. on $\partial\Omega(M,\epsilon)$.

Due to **G.Besson ('85), Chavel-D.Feldman ('88), C.Courtois ('95),** the following results have been established.

Theorem. Assume λ_k is simple in (1)

$$\lambda_k^D(\epsilon) - \lambda_k = \begin{cases} \left((n-m-2)|S^{n-m-1}| \int_M \Phi_k(\xi)^2 ds\right) \, \epsilon^{n-m-2} + \text{H.O.T.} & \text{for } n-m \geqq 3 \\ \left(2\pi \int_M \Phi_k(\xi)^2 ds\right) / \log(1/\epsilon) + \text{H.O.T.} & \text{for } n-m = 2 \end{cases}$$

Here S^{n-m-1} is the unit sphere in $\mathbb{R}^{n-m}$ and "H.O.T." implies "a higher order term".

The case of Neumann B.C., Robin B.C. on $\Gamma(M, \epsilon)$

Perturbed eigenvalue problems

$$(5) \qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \quad \text{on} \quad \Gamma, \\ \dfrac{\partial\Phi}{\partial\nu} + \sigma\epsilon^{\tau}\Phi = 0 \quad \text{on} \quad \Gamma(M, \epsilon). \qquad (<= \text{Robin B.C.}) \end{cases}$$

$$(6) \qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \quad \Phi = 0 \quad \text{on} \quad \Gamma, \\ \dfrac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \Gamma(M, \epsilon). \qquad (<= \text{ Neumann B.C.}) \end{cases}$$

Here ν is the unit outward normal vector on $\partial\Omega(\epsilon)$ and $\sigma > 0, \tau \in \mathbb{R}$ are parameters.

Eigenvalues and Eigenfunctions in $\Omega(\epsilon)$

Definition. We denote the eigenvalues of (3) by $\{\lambda_k^R(\epsilon)\}_{k=1}^{\infty}$ and the corresponding complete orthonormal system by $\{\Phi_{k,\epsilon}^R\}_{k=1}^{\infty} \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^R, \Phi_{\ell,\epsilon}^R)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

Definition. We denote the eigenvalues of (4) by $\{\lambda_k^N(\epsilon)\}_{k=1}^{\infty}$ and the corresponding complete orthonormal system $\{\Phi_{k,\epsilon}^N\}_{k=1}^{\infty} \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^N, \Phi_{\ell,\epsilon}^N)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

Proposition. For $k \in \mathbb{N}$, $\lambda_k^N(\epsilon) \leqq \lambda_k^R(\epsilon) \leqq \lambda_k^D(\epsilon) \leqq \lambda_k + o(1)$ for $\epsilon \to 0$.

(Sketch of the proof) This is proved by a (rough) test functions

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) & \text{for} \quad x \in \Omega \setminus B(M, r_0) \\ \Phi_k(x)\frac{\log(\epsilon/r)}{\log(\epsilon/r_0)} & \text{for} \quad x \in B(M, r_0) \setminus B(M, \epsilon), r = \mathrm{dist}(x, M) \end{cases}.$$

with the max-min principle through the Rayleigh quotient

$$\mathcal{R}_\epsilon(\Phi) = \|\nabla\Phi\|^2_{L^2(\Omega(\epsilon))}/\|\Phi\|^2_{L^2(\Omega(\epsilon))}$$

$$\lambda_k^D(\epsilon) = \sup_{\dim E \leqq k-1} \inf\{\mathcal{R}_\epsilon(\Phi) \mid \Phi \in H_0^1(\Omega(\epsilon)),\ \Phi \perp E \text{ in } L^2(\Omega(\epsilon))\}$$

Here E is a subspace of $L^2(\Omega(\epsilon))$.

$$\|\widetilde{\Phi}_{k,\epsilon}-\Phi_k\|^2_{L^2(\Omega(\epsilon))}=O(1/|\log\epsilon|^2),\quad \|\nabla(\widetilde{\Phi}_{k,\epsilon}-\Phi_k)\|^2_{L^2(\Omega(\epsilon))}=\begin{cases}O(1/|\log\epsilon|^2) & \text{if}\quad q\geqq 3\\ O(1/|\log\epsilon|) & \text{if}\quad q=2\end{cases}$$

$$(\widetilde{\Phi}_{k,\epsilon},\widetilde{\Phi}_{k',\epsilon})_{L^2(\Omega(\epsilon))}=\delta(k,k')+O(\frac{1}{|\log\epsilon|}),$$

$$(\nabla\widetilde{\Phi}_{k,\epsilon},\nabla\widetilde{\Phi}_{k',\epsilon})_{L^2(\Omega(\epsilon))}=\lambda_\ell\,\delta(k,k')+\begin{cases}O(\frac{1}{|\log\epsilon|^{1/2}}) & \text{for } q=2\\ O(\frac{1}{|\log\epsilon|}) & \text{for } q\geqq 3\end{cases}$$

Put $F=L.H.[\widetilde{\Phi}_{1,\epsilon},\widetilde{\Phi}_{2,\epsilon},\cdots,\widetilde{\Phi}_{k,\epsilon}]$ and see $\dim(F)=k$.
Take any subspace $E\subset L^2(\Omega(\epsilon))$ with $\dim(E)\leqq k-1$, then there exists

$$\Psi=\sum_{\ell=1}^{k}c_k\widetilde{\Phi}_{\ell,\epsilon}\in F,\quad \Psi\perp E \text{ in } L^2(\Omega(\epsilon)),\quad \sum_{\ell=1}^{k}c_\ell^2=1.$$

Then we have

$$\inf_{\Phi\in H_0^1(\Omega(\epsilon)),\Phi\perp E\,\mathrm{in}\,L^2(\Omega(\epsilon))}\mathcal{R}_\epsilon(\Phi)\leqq\mathcal{R}_\epsilon(\Psi)=\frac{\|\nabla(\sum_{\ell=1}^k c_k\widetilde{\Phi}_{\ell,\epsilon})\|^2_{L^2(\Omega(\epsilon))}}{\|\sum_{\ell=1}^k c_k\widetilde{\Phi}_{\ell,\epsilon}\|^2_{L^2(\Omega(\epsilon))}}$$

$$= \frac{\sum_{1\leqq \ell,\ell'\leqq k}(\nabla\widetilde{\Phi}_{\ell,\epsilon},\nabla\widetilde{\Phi}_{\ell',\epsilon})_{L^2(\Omega(\epsilon))}c_\ell c_{\ell'}}{\sum_{1\leqq \ell,\ell'\leqq k}(\widetilde{\Phi}_{\ell,\epsilon},\widetilde{\Phi}_{\ell',\epsilon})_{L^2(\Omega(\epsilon))}c_\ell c_{\ell'}} = \frac{\sum_{\ell=1}^k \lambda_\ell(1+o(1))c_\ell^2 + \sum_{1\leqq \ell\neq\ell'\leqq k} o(1)c_\ell c_{\ell'}}{\sum_{\ell=1}^k (1+o(1))c_\ell^2 + \sum_{1\leqq \ell\neq\ell'\leqq k} o(1)c_\ell c_{\ell'}}$$

$$\leqq \frac{\lambda_k + o(1)}{1 - k^2 o(1)} \leqq \lambda_k + o(1)$$

Note that the right hand side is independent of choice of E. Taking sup for all choices of $E \subset L^2(\Omega(\epsilon)), \dim E \leqq k-1$ with the max min principle

$$\lambda_k^D(\epsilon) \leqq \lambda_k + o(1)$$

Since $\lambda_k \leqq \lambda_k^D(\epsilon)$, $\lim_{\epsilon\to 0}\lambda_k^D(\epsilon) = \lambda_k$ follows.

Proposition (Convergence). For $k \in \mathbb{N}$, we have

$$\lim_{\epsilon \to 0} \lambda_k^R(\epsilon) = \lambda_k \quad (k \in \mathbb{N}), \quad \lim_{\epsilon \to 0} \lambda_k^N(\epsilon) = \lambda_k \quad (k \in \mathbb{N}).$$

Proposition (Uniform bound). For each $k \in \mathbb{N}$, there exist $\epsilon_0 > 0$ and $c(k) > 0$ such that

$$|\Phi_{k,\epsilon}^R(x)| \leqq c(k), \quad |\Phi_{k,\epsilon}^N(x)| \leqq c(k) \quad (x \in \Omega(\epsilon),\ 0 < \epsilon \leqq \epsilon_0).$$

Notation

∇ : the gradient in $\mathbb{R}^n$
∇_M : the tangential gradient on M
∇_N : the normal gradient at a point of the manifold M

$$\nabla\phi = \nabla_M\phi + \nabla_N\phi \quad \text{on} \quad M$$

Notation

Denote the **mean curvature vector** field on M by H. H is a normal vector field on M. As an operator, for a function ϕ defined in a neighborhood of M, H acts on ϕ as a differential in H direction as follows

$$H[\phi](\xi) = \lim_{t\to 0}(\phi(\xi + tH(\xi)) - \phi(\xi))/t \quad \text{at each} \quad \xi \in M.$$

Theorem. Assume that $n-m=q\geqq 3$ and λ_k is simple in (1).
(0) We have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^N(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_kH[\Phi_k]\right\}ds(\xi).$$

(i) Assume $\tau>1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_kH[\Phi_k]\right\}ds(\xi).$$

(ii) Assume $\tau=1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+(\lambda_k+q\sigma)\Phi_k^2-\Phi_kH[\Phi_k]\right\}ds(\xi).$$

(iii) Assume $-1 < \tau < 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q+\tau-1}} = \sigma |S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

(iv) Assume $\tau = -1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q-2}} = \frac{\sigma(q-2)}{q-2+\sigma} |S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

(v) Assume $\tau < -1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{q-2}} = (q-2) |S^{q-1}| \int_M \Phi_k(\xi)^2 ds(\xi).$$

Here $|S^{q-1}| = 2\pi^{q/2}/\mathbf{\Gamma}(q/2)$, which is is the measure of S^{q-1} and $\mathbf{\Gamma}(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$ is the Gamma function.

Theorem. Assume that $n - m = q = 2$ and λ_k is simple in (1).
(0) We have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^N(\epsilon) - \lambda_k}{\epsilon^2} = \pi \int_M \left(-2|\nabla_N \Phi_k|^2 - |\nabla_M \Phi_k|^2 + \lambda_k \Phi_k^2 - \Phi_k H[\Phi_k]\right) ds(\xi).$$

(i) Assume $\tau > 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^2} = \pi \int_M \left(-2|\nabla_N \Phi_k|^2 - |\nabla_M \Phi_k|^2 + \lambda_k \Phi_k^2 - \Phi_k H[\Phi_k]\right) ds(\xi).$$

(ii) Assume $\tau = 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^2} = \pi \int_M \left(-2|\nabla_N \Phi_k|^2 - |\nabla_M \Phi_k|^2 + (\lambda_k + 2\sigma) \Phi_k^2 - \Phi_k H[\Phi_k]\right) ds(\xi).$$

(iii) Assume $-1 < \tau < 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{1+\tau}} = 2\pi\sigma \int_M \Phi_k(\xi)^2 ds(\xi).$$

(iv) Assume $\tau \leqq -1$, then we have

$$\lim_{\epsilon \to 0} (\lambda_k^R(\epsilon) - \lambda_k) \log(1/\epsilon) = 2\pi \int_M \Phi_k(\xi)^2 ds(\xi).$$

Remark. It should be noted that in the case $\tau < -1$ in Theorem 3 and Theorem 4, the formula takes the same form as $\lambda_k^D(\epsilon)$ (the case of the Dirichlet B.C. on $\Gamma(M, \epsilon)$). In this case the Robin B.C. is close to the Dirichlet B.C. On the other hand, the formulas for $\lambda_k^R(\epsilon)$ for $\tau > 1$ (in (i)) takes the same form as $\lambda_k^N(\epsilon)$ (in (0)).

Remark. S. Ozawa dealt with $n = 3$, $\dim M = 1$ and proved (iii) in Theorem 4 with other method in his preprint: S. Ozawa, Spectra of the Laplacian and singular variation of domain - removing an $\epsilon-$ neighborhood of a curve, unpublished note (1998).

Sketch of the proof

[I] Characterization of the eigenfunction $\Phi^{R}_{k,\epsilon}(x)$, $\Phi^{N}_{k,\epsilon}(x)$
Estimates for uniform bound and convergence

[II] Construction of the approximate eigenfunction $\widetilde{\Phi}^{R}_{k,\epsilon}(x)$, $\widetilde{\Phi}^{N}_{k,\epsilon}(x)$
Explicit expression of the approximation

[Coordinate system in $B(M, r_0)$]

M : a compact $m-$dimensional smooth manifold (M has a finite covering)

$$\mathbb{R}^n = T_\xi M \oplus N_\xi M \quad (\xi \in M) \qquad \text{(orthogonal decomposition)}$$

Here $\dim(T_\xi M) = m$, $\dim(N_\xi M) = q$.

Let $(e_1(\xi), e_2(\xi), \cdots, e_q(\xi))$ be an orthonormal frame in $N_\xi M$ (smooth in ξ) in a chart of the covering. $\exists r_0 > 0$ such that

$$B(M, r_0) \ni x = \xi + \sum_{\ell=1}^{q} \eta_\ell \, e_\ell(\xi).$$

Denote the second term by $\eta \cdot e(\xi)$.

[**Mean curvature operator (vector) on** M]

The second fundamental form $h_\xi(X,Y)$ of M is defined by the following formula

$$\nabla_Y X = \nabla_Y^M X + h_\xi(X,Y) \in T_\xi M \oplus N_\xi M \quad \text{(orthogonal decomposition)}$$

for any C^1 vector fields X, Y which are defined in a neighborhood of M and tangent to M. The mean curvature vector H of M is defined by

$$H_\xi = \sum_{i=1}^{m} h_\xi(E_i, E_i)$$

for each $\xi \in M$. Here $\{E_1, E_2, \cdots, E_m\}$ is an orthonormal frame of $T_\xi M$ (cf. Kobayashi-Nomizu ('63)).

Lemma. In this coordinate system $(\xi, \eta) = (\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$, the mean curvature operator of M is expressed as follows.

$$H_\xi = -\sum_{\ell=1}^{q} \frac{1}{\sqrt{g(\xi,0)}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{|M} \frac{\partial}{\partial \eta_\ell} = -\sum_{\ell=1}^{q} \sum_{1 \leqq i,j \leqq m} \frac{g^{ij}(\xi,0)}{2} \frac{\partial g_{ij}}{\partial \eta_\ell}(\xi,0) \frac{\partial}{\partial \eta_\ell}$$

It is also expressed as a normal vector field

$$H_\xi = -\sum_{\ell=1}^{q} \frac{1}{\sqrt{g(\xi,\mathbf{0})}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{\eta=\mathbf{0}} e_\ell(\xi).$$

Proposition. For a C^2 function u which is defined in $B(M, r_0)$, we have

$$(\Delta u)_{|M} = \Delta_M(u_{|M}) - H[u] + \sum_{\ell=1}^{q} \left(\frac{\partial^2 u}{\partial \eta_\ell^2} \right)_{|\eta=\mathbf{0}} \quad \text{on} \quad M.$$

[I] Uniform bound for the eigenfunction $\Phi^R_{k,\epsilon}$, $\Phi^N_{k,\epsilon}$

Lemma (Barrier function). There exsits a function $\psi_\epsilon(x)$ (defined from K_1) satisfies the following properties. For any $m_2 > 0$, there exist $\epsilon_1 > 0$, $r_1 \in (0, r_0]$ and $\epsilon_1 > 0$ such that

$$\Delta\psi_\epsilon + m_2\psi_\epsilon \leqq 0 \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon),$$

$$\frac{\partial\psi_\epsilon}{\partial\nu} \geqq 0 \quad \text{on} \quad \Gamma(M, \epsilon), \quad 1 \leqq \psi_\epsilon(x) \leqq 3 \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon),$$

for any $\epsilon \in (0, \epsilon_1)$.

Estimates for $\Phi^R_{k,\epsilon}$, $\Phi^N_{k,\epsilon}$

For any $r_1 > 0$, there exists $c > 0$ such that $|\Phi^R_{k,\epsilon}(x)| \leqq c$ in $\Omega \setminus B(M, r_1)$ and $0 < \epsilon \leqq r_1/2$ (Elliptic estimates).

By the comparison argument, we have

$$-c\,\psi_\epsilon(x) \leqq \Phi^R_{k,\epsilon}(x) \leqq c\,\psi_\epsilon(x) \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon).$$

for $\epsilon > 0$.

Same argument applies to $\Phi^N_{k,\epsilon}$.

[II] Approximate eigenfunction $\widetilde{\Phi}^R_{k,\epsilon}$

We first construct an approximate eigenfunction $\widetilde{\Phi}_{k,\epsilon}$, by modifying Φ_k around M according to the Robin B.C. on $\Gamma(M,\epsilon)$. We consider $\phi(\eta)=\phi(\eta_1,\cdots,\eta_q)$ satisfying

$$\begin{cases} \Delta_\eta \phi = 0 \quad \text{for} \quad \epsilon < |\eta| < r_0, \quad \phi = 0 \quad \text{for} \quad |\eta| = r_0, \\ \left(\dfrac{\partial \phi}{\partial \nu_\eta} + \sigma\epsilon^\tau \phi\right)_{|\eta|=\epsilon} = \left(\dfrac{\partial}{\partial \nu_\eta}\Phi_k(\xi+\eta\cdot\boldsymbol{e}(\xi)) + \sigma\epsilon^\tau\Phi_k(\xi+\eta\cdot\boldsymbol{e}(\xi))\right)_{|\eta|=\epsilon} \end{cases}$$

for each $\xi \in M$. Here $\Delta_\eta = \partial^2/\partial\eta_1^2 + \cdots + \partial^2/\partial\eta_q^2$. Basic harmonic functions in η space solutions are given by

$$r^\ell \varphi_{\ell,p}(\omega), \quad r^{-\ell-q+2}\varphi_{\ell,p}(\omega) \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)) \quad \text{harmonic functions in} \quad \mathbb{R}^q \setminus \{\mathbf{0}\}.$$

where $\{\varphi_{\ell,p}(\omega)\}_{\ell\geqq 0, 1\leqq p \leqq \iota(\ell)}$ are eigenfunctions of the Laplace-Beltrami operator in S^{q-1}. The eigenvalues $\gamma(\ell)$ and its multiplicity $\iota(\ell)$ are given as follows

$$\gamma(\ell) = \ell(\ell+q-2), \;\; \iota(\ell) = \frac{(2\ell+q-2)(q+\ell-3)!}{(q-2)!\ell!} \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)$$

The solution of the Laplace equation

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{l,p} r^{\ell} + b_{\ell,p} r^{-\ell-q+2}) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

The coefficients $a_{\ell,p}$, $b_{\ell,p}$ can be calculated by the infinite series of relations determined by the boundary condition. From the boundary condition on $|\eta| = r_0$, we have

$$\sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{\ell,p} r_0^{\ell} + b_{\ell,p} r_0^{-\ell-q+2}) \varphi_{\ell,p}(\omega) = 0 \quad (\omega \in S^{q-1})$$

which gives

$$a_{\ell,p} r_0^{\ell} + b_{\ell,p} r_0^{-\ell-q+2} = 0 \quad \text{for} \quad \ell \geqq 0, 1 \leqq p \leqq \iota(\ell).$$

ϕ is written by

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} (r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell}) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

We calculate the Robin condition on $|\eta| = \epsilon$. Noting

$$\frac{\partial}{\partial\nu} = -\frac{\partial}{\partial r} = -\sum_{i=1}^{q} \frac{\eta_i}{|\eta|}\frac{\partial}{\partial\eta_i} \quad \text{on} \quad \Gamma(M,\epsilon) = \{x = \xi + \eta\cdot e(\xi) \mid \xi \in M, |\eta| = \epsilon\}$$

we have the equations for the coefficients $a_{\ell,p}, b_{\ell,p}$ as follows.

$$\begin{aligned}
&-\sum_{\ell\geqq 0, 1\leqq p\leqq\iota(\ell)} b_{\ell,p}\left((-\ell-q+2)r^{-\ell-q+1} - \ell r_0^{-2\ell-q+2} r^{\ell-1}\right)_{r=\epsilon} \varphi_{\ell,p}(\omega)\\
&\qquad + \sum_{\ell\geqq 0, 1\leqq p\leqq\iota(\ell)} b_{\ell,p}\sigma\epsilon^{\tau}\left(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell}\right)_{r=\epsilon} \varphi_{\ell,p}(\omega)\\
&= -\sum_{i=1}^{q} \langle \nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi)\rangle \eta_i/|\eta| + \sigma\epsilon^{\tau}\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi))
\end{aligned}$$

for $\omega \in S^{q-1}$. Multiply both sides by $\varphi_{p,\ell}$ and integrate on S^{q-1} and we get

$$\begin{aligned}
&b_{\ell,p}\left\{(\ell+q-2)\epsilon^{-\ell-q+1} + \ell r_0^{-2\ell-q+2}\epsilon^{\ell-1} + \sigma(\epsilon^{-\ell-q+2+\tau} - r_0^{-2\ell-q+2}\epsilon^{\ell+\tau})\right\}\\
&= \int_{S^{q-1}} \left\{-\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\} + \sigma\epsilon^{\tau}\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi))\right\} \varphi_{\ell,p}(\omega)d\omega
\end{aligned}$$

From these equations we get $a_{\ell,p}, b_{\ell,p}$ as follows

$$a_{\ell,p} = -r_0^{-2\ell-q+2} b_{\ell,p}$$

$$b_{\ell,p} = \frac{1}{(\ell+q-2)\epsilon^{-\ell-q+1} + \ell r_0^{-2\ell-q+2}\epsilon^{\ell-1} + \sigma(\epsilon^{-\ell-q+2+\tau} - r_0^{-2\ell-q+2}\epsilon^{\ell+\tau})}$$

$$\times \int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\} + \sigma\epsilon^{\tau}\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)) \right\} \varphi_{\ell,p}(\omega) d\omega$$

We remark that these ($\epsilon-$dependent) coefficients $a_{\ell,p}$, $b_{\ell,p}$ are smoothly dependent on $\xi \in M$ since Φ_k is smooth. So we denote this function $\phi(x)$ in $B(M, r_0) \setminus B(M, \epsilon)$ by $G_{k,\epsilon}(x)$. That is

$$G_{k,\epsilon}(x) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p}(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell})\varphi_{\ell,p}(\omega) \quad (x = \xi + (r\omega)\cdot e(\xi)).$$

Definition. The approximate eigenfunction is defined by

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) & \text{for} \quad x \in \Omega \setminus B(M, r_0) \\ \Phi_k(x) - G_{k,\epsilon}(x) & \text{for} \quad x = \xi + \eta\cdot e(\xi) \in B(M, r_0) \setminus B(M, \epsilon) \end{cases}$$

Lemma. (i) $\ell = 0$

$$b_{0,1} = |S^{q-1}|^{1/2} \begin{cases} \frac{-\epsilon^q}{q(q-2)}\{\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + O(\epsilon)\} & (\tau > 1) \\ \frac{\epsilon^q}{q-2}\{(-1/q)\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + \sigma\Phi_k(\xi) + O(\epsilon)\} & (\tau = 1) \\ \frac{\sigma\epsilon^{q-1+\tau}}{q-2}(\Phi_k(\xi) + O(\epsilon)) & (-1 < \tau < 1) \\ \frac{\sigma\epsilon^{q-2}}{q-2+\sigma}(\Phi_k(\xi) + O(\epsilon)) & (\tau = -1) \\ \epsilon^{q-2}(\Phi_k(\xi) + O(\epsilon)) & (\tau < -1) \end{cases}$$

(ii) $\ell = 1$

$$b_{1,p} = \frac{|S^{q-1}|^{1/2}}{q^{1/2}} \langle \nabla\Phi_k(\xi), e_p(\xi)\rangle \epsilon^q (1 + O(\epsilon)) \times \begin{cases} -1/(q-1) & (\tau > -1) \\ (\sigma - 1)/(q - 1 + \sigma) & (\tau = -1) \\ 1 & (\tau < -1) \end{cases}$$

Lemma. For any $N \in \mathbb{N}$, there exists $d_N > 0$ (independent of $\xi \in M$) such that

$$|b_{\ell,p}| \leqq \frac{d_N}{\gamma(\ell)^N} \begin{cases} \epsilon^{\ell+q} & (\tau \geqq 0) \\ \epsilon^{\ell+q+\tau} & (-1 < \tau < 0)\,, \quad (1 \leqq p \leqq \iota(\ell), \ell \geqq 2). \\ \epsilon^{\ell+q-1} & (\tau \leqq -1) \end{cases}$$

Proof for the theorems

For any sequence of positive values $\{\epsilon_p\}_{p=1}^\infty$ with $\lim_{p\to\infty}\epsilon_p = 0$, there exists a subsequence $\{\zeta_p\}_{p=1}^\infty$ and orthonormal systems of eigenfunctions $\{\Phi'_k\}_{k=1}^\infty$ and $\{\Phi''_k\}_{k=1}^\infty$ of (1) corresponding to $\{\lambda_k\}_{k=1}^\infty$, respectively such that

$$(\Phi'_k, \Phi'_\ell)_{L^2(\Omega)} = \delta(k,\ell), \quad (\Phi''_k, \Phi''_\ell)_{L^2(\Omega)} = \delta(k,\ell) \quad (k,\ell\in\mathbb{N}),$$

$$\lim_{p\to\infty}\|\Phi^R_{k,\zeta_p} - \Phi'_k\|_{L^2(\Omega(\zeta_p))} = 0, \quad \lim_{p\to\infty}\|\Phi^N_{k,\zeta_p} - \Phi''_k\|_{L^2(\Omega(\zeta_p))} = 0.$$

Calculation of the limit behavior of $\lambda_k^R(\epsilon) - \lambda_k$.

$$(7) \qquad \int_{\Omega(\epsilon)} (\Delta \Phi_{k,\epsilon}^R + \lambda_k^R(\epsilon)\Phi_{k,\epsilon}^R)\widetilde{\Phi}_{k,\epsilon} dx = 0$$

Assume the situation $\Phi_{k,\epsilon}^R \longrightarrow \Phi_k'$ for $\epsilon = \zeta_p \to 0$ as in Proposition 2.

Calculation on the above integral relation gives

$$(8) \qquad (\lambda_k(\epsilon) - \lambda_k) \int_{\Omega(\epsilon)} \Phi_k(x)\Phi_{k,\epsilon}(x)dx = I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon).$$

where

$$I_1(\epsilon) = -\int_{\Gamma(M,r_0)} \frac{\partial G_{k,\epsilon}}{\partial \nu_1}(\Phi_{k,\epsilon}(x) - \Phi_k'(x))dS$$

$$I_2(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} G_{k,\epsilon}(x)(\Delta\Phi_k'(x) + \lambda_k(\epsilon)\Phi_{k,\epsilon}(x))dx$$

$$I_3(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} (\Delta G_{k,\epsilon}(x))(\Phi_{k,\epsilon}(x) - \Phi_k'(x))dx$$

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left(\frac{\partial G_{k,\epsilon}}{\partial \nu_1} \Phi_k' - G_{k,\epsilon} \frac{\partial \Phi_k'}{\partial \nu_1} \right) dS$$

$I_4(\epsilon)$ is also written

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left((\frac{\partial \Phi_k}{\partial \nu_1} + \sigma \epsilon^\tau \Phi_k) \Phi_k' - G_{k,\epsilon} (\frac{\partial \Phi_k'}{\partial \nu_1} + \sigma \epsilon^\tau \Phi_k') \right) dS.$$

Careful evaluation on I_1, I_2, I_3, I_4 gives the perturbation formula in Theorem.

(B) Domain with partial degeneration

$D \subset \mathbb{R}^n$: a bounded domain (or a finite union of bounded domains) with a smooth boundary. The perturbed domain

$$\Omega(\zeta) = D \cup Q(\zeta) \subset \mathbb{R}^n$$

Here $Q(\zeta)$ is a thin set which approaches a lower dimensional set L as $\zeta \to 0$.

Eigenvalue problem

$$\Delta\Phi + \mu\Phi = 0 \quad \text{in} \quad \Omega(\zeta), \qquad \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega(\zeta) \tag{9}$$

Let $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ be the eigenvalues with the corresponding eigenfunctions $\Phi_{k,\zeta}$ $(k \geqq 1)$ such that

$$(\Phi_{k,\zeta}, \Phi_{k',\zeta})_{L^2(\Omega(\zeta))} = \delta(k, k') \quad (k, k' \geqq 1) \quad \text{(Kronecker's delta)}$$

Basic question : What is the limitting behavior of $\mu_k(\zeta)$ for $\zeta \to 0$?

For the Dumbbell domain, there are results. Beale('73), Fang('93), Jimbo('93), Gadylshin('93), Arrieta('95), Jimbo-Morita('95), Anné('87),...

Convergence of the eigenvalues. Perturbation formula (first order approximation) is studied.

In this lecture I deal with more genral cases. Hereafter I mainly talk about the results in Jimbo-Kosugi('09).

The construction of $\Omega(\zeta) = D \cup Q(\zeta)$

Let $n, \ell, m \in \mathbb{N}$ with $n = \ell + m$. $x = (x', x'') \in \mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$
$D \subset \mathbb{R}^n$, $L \subset \mathbb{R}^\ell$ bounded domains (finite disjoint union of bounded domains) with smooth boundaries.

Some symbols:

$$B^{(m)}(s) := \{x'' \in \mathbb{R}^m \mid |x''| < s\}, \quad L(s) := \{x' \in L \mid \mathrm{dist}(x', \partial L) > s\}$$

Assumption: There exists $\zeta_0 > 0$ such that

$$(\overline{L} \times B^{(m)}(3\zeta_0)) \cap D = \partial L \times B^{(m)}(3\zeta_0) \subset \partial D$$

There exists a function $\rho = \rho(t) \in C^3((-\infty, 0)) \cap C^0((-\infty, 0])$ such that

$$\rho(t) = 1 \ (t \leqq -1), \ \rho'(t) > 0 \ (-1 < t < 0), \ \rho(0) = 2, \ \lim_{s \uparrow 2} d^k \rho^{-1}(s)/ds^k = 0 \quad (1 \leqq k \leqq 3)$$

Put $Q(\zeta) = Q_1(\zeta) \cup Q_2(\zeta)$ where $Q_1(\zeta) = L(2\zeta) \times B^{(m)}(\zeta)$ and

$$Q_2(\zeta) = \{(\xi + s\nu'(\xi), \eta) \mid \mathbb{R}^\ell \times \mathbb{R}^m \mid -2\zeta \leqq s \leqq 0, \xi \in \partial L, |\eta| < \zeta\rho(s/\zeta)\}$$

To express the limit of $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ we prepare the notation.

Definition. $\{\omega_k\}_{k=1}^{\infty}$ is the system of eigenvalues of

(10) $$\Delta\phi + \omega\phi = 0 \text{ in } D, \quad \partial\phi/\partial\nu = 0 \text{ on } \partial D$$

Definition. $\{\lambda_k\}_{k=1}^{\infty}$ is the system of eigenvalues of

(11) $$\Delta'\psi + \lambda\psi = 0 \text{ in } L, \quad \psi = 0 \text{ on } \partial L$$

where

$$\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_\ell^2}$$

The limit of $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ is given by the following result.

Proposition. $\lim_{\zeta\to 0}\mu_k(\zeta) = \mu_k$ for any $k \geqq 1$ where $\{\mu_k\}_{k=1}^{\infty}$ is given by rearranging $\{\omega_k\}_{k=1}^{\infty} \cup \{\lambda_k\}_{k=1}^{\infty}$ in increasing order with counting multiplicity.

Remark. μ_k is written as

$$\mu_k = \max_{1\leqq j\leqq k}\left(\min(\omega_{k+1-j}, \lambda_j)\right).$$

Classification of eigenvalues

Definition

$$E_I = \{\omega_k\}_{k=1}^\infty \setminus \{\lambda_k\}_{k=1}^\infty, \qquad E_{II} = \{\lambda_k\}_{k=1}^\infty \setminus \{\omega_k\}_{k=1}^\infty, \qquad E_{III} = \{\omega_k\}_{k=1}^\infty \cap \{\lambda_k\}_{k=1}^\infty$$

Relation to the eigenfunctions

Let $\{\Phi_{k,\zeta}\}_{k=1}^\infty \subset L^2(\Omega(\zeta))$ be the (complete) orthonormal system corresponding to $\{\mu_k(\zeta)\}_{k=1}^\infty$ of (9).

Proposition.

$$\lim_{\zeta\to 0} \|\Phi_{k,\zeta}\|_{L^2(Q(\zeta))} = 0 \iff \mu_k \in E_I$$

$$\lim_{\zeta\to 0} \|\Phi_{k,\zeta}\|_{L^2(D)} = 0 \iff \mu_k \in E_{II}$$

$$\liminf_{\zeta\to 0} \|\Phi_{k,\zeta}\|_{L^2(Q(\zeta))} > 0,\ \liminf_{\zeta\to 0} \|\Phi_{k,\zeta}\|_{L^2(D)} > 0 \iff \mu_k \in E_{III}$$

Proposition (Convergence rate)

$$\mu_k \in E_I \Longrightarrow \mu_k(\zeta) - \mu_k = O(\zeta^m)$$

$$\mu_k \in E_{II} \Longrightarrow \mu_k(\zeta) - \mu_k = \begin{cases} O(\zeta) & (m \geqq 2) \\ O(\zeta \log(1/\zeta)) & (m = 1) \end{cases}$$

For $\mu_k \in E_{III}$, a mixed situation occurs (as seen later).

Some preparation(uniform converegence)

Consider the following semilinear elliptic equation in $\Omega(\zeta)$.

$$\Delta u + f_\zeta(u) = 0 \quad \text{in} \quad \Omega(\zeta), \quad \partial u/\partial\nu = 0 \quad \text{on}$$

Here $\zeta > 0$ is a parameter and the nonlinear term $f_\zeta(u)$ is assumed to be a C^1 function in $\mathbb{R}$ such that $(\partial f_\zeta/\partial u)(u)$ is uniformly bounded in $\mathbb{R}$ and $f_\zeta(u)$ converges locally uniformly to a C^1 function $f_0(u)$ for $\zeta \to 0$.

Theorem ([8]). Let $\{\zeta_p\}_{p=1}^{\infty}$ be a positive sequence which converges to 0 as $p \to \infty$ and let $u_{\zeta_p} \in C^2(\overline{\Omega(\zeta_p)})$ be a solution of the above equation for $\zeta = \zeta_p$ such that

$$\sup_{p \geqq 1} \sup_{x \in \Omega(\zeta_p)} |u_{\zeta_p}(x)| < \infty.$$

Then there exists a subsequence $\{\sigma_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$ and functions $w \in C^2(\overline{D})$ and $V \in C^2(\overline{L})$ such that

$$\Delta w + f_0(w) = 0 \quad \text{in} \quad D, \; \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial D \quad \text{(Neumann B.C.)},$$
$$\Delta' V + f_0(V) = 0 \quad \text{in} \quad L, \; V(x') = w(x', o'') \quad \text{for} \quad x' \in \partial L,$$
$$\lim_{p\to\infty} \sup_{x\in D} |u_{\sigma_p}(x) - w(x)| = 0,$$
$$\lim_{p\to\infty} \sup_{(x',x'')\in Q(\sigma_p)} |u_{\sigma_p}(x', x'') - V(x')| = 0,$$

where $\Delta' = \sum_{k=1}^{\ell} \partial^2/\partial x_k^2$. Note that $\partial L \times \{o''\} \subset \partial D$.

Perturbation formula [Type (I)]

Let $\{\phi_k\}_{k=1}^{\infty}$ be the system of eigenfunctions of (10) (eigenvalue problem in D) orthonormalized in $L^2(D)$.

Assume $\mu_k \in E_I$ and there exists $k' \in \mathbb{N}$ such that $\mu_k = \omega_{k'}$. Assume also that $\omega_{k'}$ is a simple eigenvalue of (10).

Theorem.

$$\mu_k(\zeta) - \mu_k = S(m)\alpha(k)\zeta^m + o(\zeta^m)$$

where

$$\alpha(k) = \int_{\partial L} \frac{\partial V_{k'}}{\partial \nu'}(\xi)\phi_{k'}(\xi, o'')dS'$$

$V_{k'}(x')$ is the unique solution $V \in C^2(\overline{L})$ of

$$\Delta' V + \omega_{k'} V = 0 \text{ in } L, \quad V(\xi) = \phi_{k'}(\xi, o'') \text{ for } \xi \in \partial L.$$

$S(m)$ is the m-dimensional volume of the unit ball in $\mathbb{R}^m$.

Perturbation formula [Type (II)]

Let $\{\psi_k\}_{k=1}^{\infty}$ be the system of eigenfunctions of (11) (eigenvalue problem in L) orthonormalized in $L^2(L)$.

Assume $\mu_k \in E_{II}$ and there exists $k'' \in \mathbb{N}$ such that $\mu_k = \lambda_{k''}$. Assume also that $\lambda_{k''}$ is a simple eigenvalue of (11).

Theorem.

$$\mu_k(\zeta) - \mu_k = -\frac{2}{\pi}\beta(k'')\zeta\log(1/\zeta) + o(\zeta\log(1/\zeta)) \quad (m = 1),$$

$$\mu_k(\zeta) - \mu_k = -T(\rho, m)\beta(k'')\zeta + o(\zeta) \quad (m \geqq 2).$$

where

$$\beta(k'') = \int_{\partial L}\left(\frac{\partial\psi_{k''}}{\partial\nu'}(\xi)\right)^2 dS'$$

and $T(\rho, m)$ is the number which depends on $\Omega(\zeta)$ (to be explained later).

Remark. For the case of Dumbbell, Gadylshin('93) obtained this result $m = 2$ and Arrieta ('95) obtained this result for $m = 1$.

Quantity $T(\rho, m)$ $(m \geqq 2)$

Harmonic function G in the set $H = H_1 \cup H_2 \subset \mathbb{R} \times \mathbb{R}^m$ where H_1, H_2 are given

$$H_1 = (0, \infty) \times \mathbb{R}^m, \qquad H_2 = \{(s, \eta) \in \mathbb{R} \times \mathbb{R}^m \mid |\eta| < \rho(s),\ s \leqq 0\}.$$

Proposition. There exists a solution G to

$$\frac{\partial^2 G}{\partial s^2} + \sum_{j=1}^{m} \frac{\partial^2 G}{\partial \eta_j^2} = 0 \quad ((s, \eta) \in H) \qquad \frac{\partial G}{\partial \boldsymbol{n}} = 0 \quad ((s, \eta) \in \partial H)$$

such that

$$G(z) = G(s, \eta) \longrightarrow 0 \quad \text{for} \quad (z \in H_1, |z| \to \infty)$$
$$G(s, \eta) - (-\kappa_1 s + \kappa_2) \longrightarrow 0 \quad \text{for} \quad (z \in H_2, |z| \to \infty)$$

where $\kappa_1 > 0, \kappa_2$ are real constants. κ_2/κ_1 is uniquely determined by H.

Definition. $T(\rho, m) = \kappa_2/\kappa_1$.

Perturbation formula [Type (III)]

Assume $\mu_k \in E_{III}$ and there exists $k', k'' \in \mathbb{N}$ such that $\mu_k = \omega_{k'} = \lambda_{k''}$. Assume also that $\omega_{k'}$ is simple eigenvalue of (10) and $\lambda_{k''}$ is a simple eigenvalue of (11).

We have the situation

$$\mu_{k-1} < \mu_k = \mu_{k+1} < \mu_{k+2}.$$

Theorem. For $m = 1$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_1^-(k', k'')\zeta^{1/2} + o(\zeta^{1/2})$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_1^+(k', k'')\zeta^{1/2} + o(\zeta^{1/2})$$

where $\gamma_1^{\pm}(k', k'')$ are eigenvalues of

$$\begin{pmatrix} 0 & \sqrt{2}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' \\ \sqrt{2}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' & 0 \end{pmatrix}$$

Theorem. For $m = 2$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_2^-(k', k'')\zeta + o(\zeta)$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_2^+(k', k'')\zeta + o(\zeta)$$

where $\gamma_2^\pm(k', k'')$ are eigenvalues of

$$\begin{pmatrix} 0 & \sqrt{\pi}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' \\ \sqrt{\pi}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' & -T(\rho, 2)\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi))^2 dS' \end{pmatrix}$$

Remark. For the case of Dumbbell ($m = 2, n = 3$), Gadylshin ('05) got this result. See Jimbo-Kosugi('09) for more genral cases.

Theorem. Assume $T(\rho, m) > 0$. For $m \geqq 3$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_m^-(k', k'')\zeta + o(\zeta)$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_m^+(k', k'')\zeta^{m-1} + o(\zeta)$$

where

$$\gamma_m^-(k', k'') = -T(\rho, m) \int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu'}(\xi) \right)^2 dS'$$

$$\gamma_m^+(k', k'') = S(m)T(\rho, m)^{-1} \left(\int_{\partial L} (\frac{\partial \psi_{k''}}{\partial \nu'}(\xi))^2 dS' \right)^{-1} \left(\int_{\partial L} \frac{\partial \psi_{k''}}{\partial \nu'}(\xi) \phi_{k'}(\xi, o'') dS' \right)^2$$

In the case $T(\rho, m) < 0$, the right hand sides are exchanged.

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