Two-dimensional V-form front

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$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u), & \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,\ t > 0 \\ u(x, y, 0) = u_0(x, y), & \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \end{cases}$$

We denote this solution by $u(x, y, t; u_0)$.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$f(u) = -u(u+a)(u-1), \quad 0 < a < \frac{1}{2}$$

$$f(0) = 0, \quad f(1) = 0$$

$$f'(0) < 0, \quad f'(1) < 0$$

$$\int_0^1 f(u)\,du > 0$$

The profile equation of one-dim traveling front

$$\Phi''(x) + k\,\Phi'(x) + f(\Phi(x)) = 0, \quad x \in \mathbb{R}$$

k : the speed of one-dim traveling front

One dimensional traveling front

$$k = \frac{1-2a}{\sqrt{2}}$$

$$\Phi(x) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right)$$

(Huxley's traveling front)

$$\Phi(x) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}$$

$$\Phi'(x) < 0, \quad x \in \mathbb{R}$$

$c \in (k, \infty)$ arbitrarily given

speed of 2-dim V-form front

$z = y - ct, \quad u(x, y, t) = w(x, z, t)$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^2 w}{\partial z^2} - c\dfrac{\partial w}{\partial z} - f(w) = 0, & \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^2, t > 0 \\ w(x, z, 0) = u_0(x, z), & \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^2 \end{cases}$$

We write $w(x, z, t)$ as $w(x, y, t)$.

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + c\dfrac{\partial w}{\partial y} + f(w), & \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, t > 0 \\ w(x, y, 0) = u_0(x, y), & \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \end{cases}$$

We denote this solution by $w(x, y, t; u_0)$.

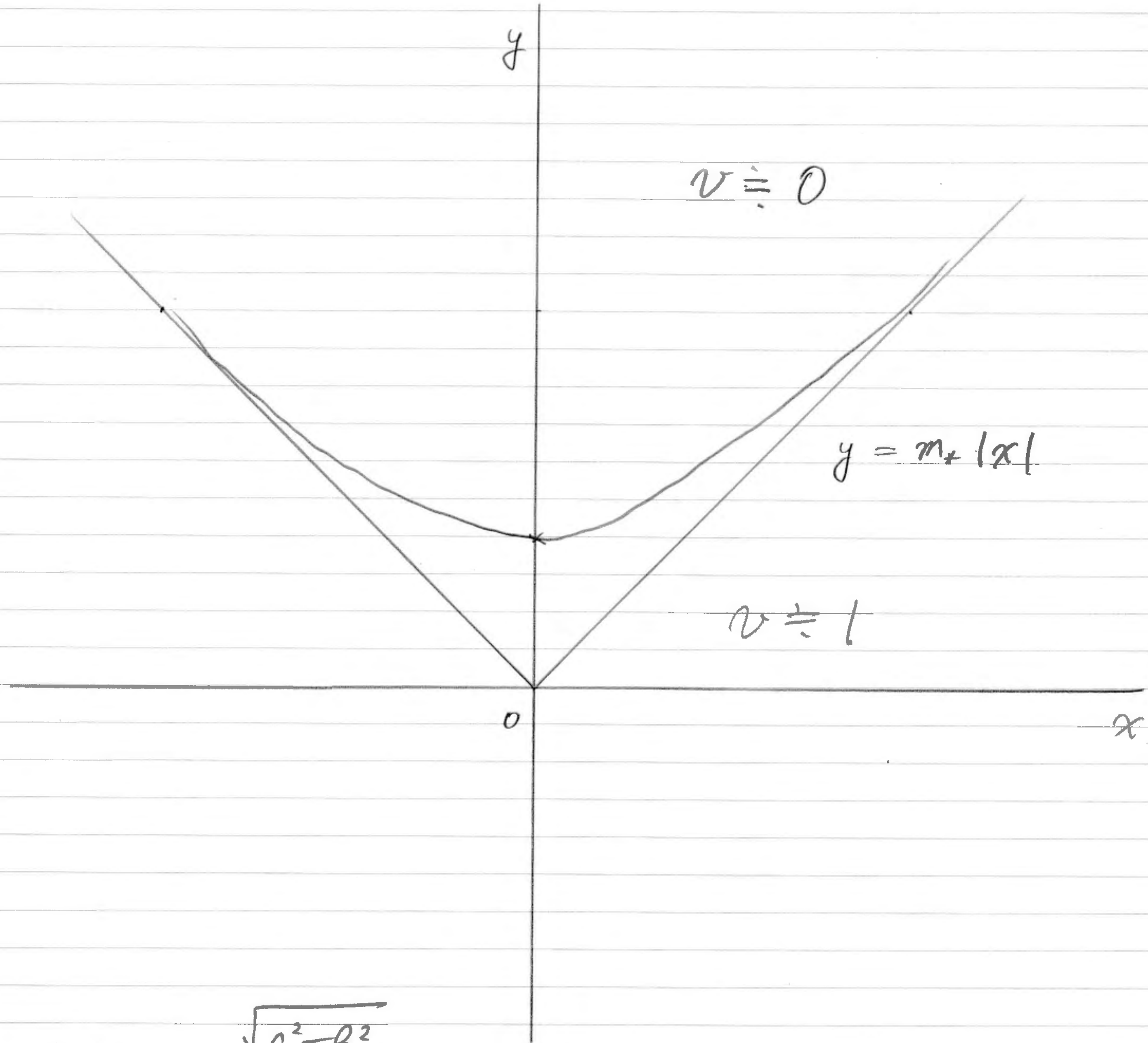
$$\mathcal{L}[w] = \frac{\partial w}{\partial t} - \Delta w - c\frac{\partial w}{\partial y} - f(w)$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The profile equation of 2-dim V-form front

$$\Delta v + c\frac{\partial v}{\partial y} + f(v) = 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$L[v] = -\Delta v - c\frac{\partial v}{\partial y} - f(v)$$

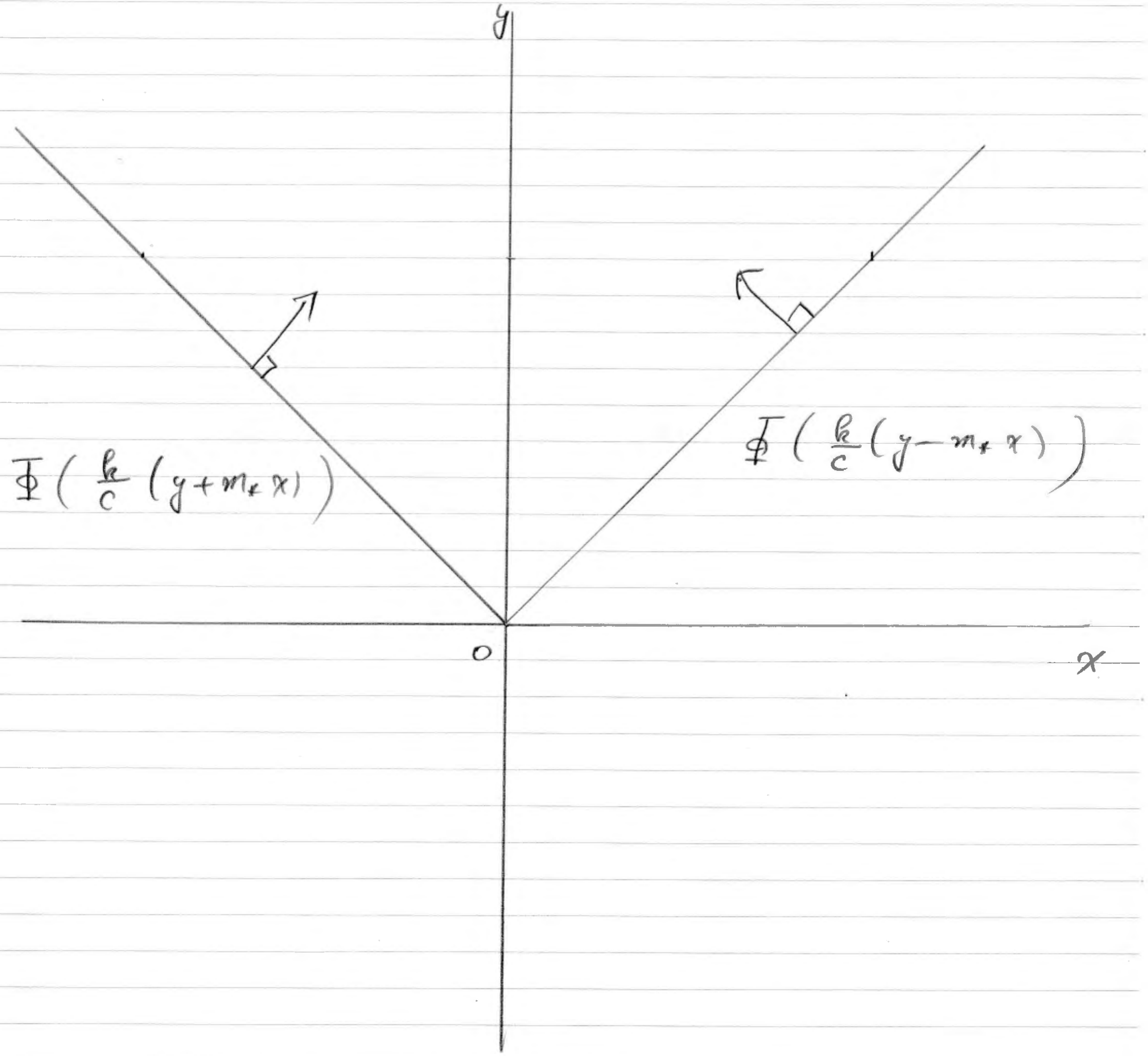


$$m_* = \frac{\sqrt{c^2 - k^2}}{k} > 0$$

Planar traveling fronts

$$v(x, y) = \Phi\left(\frac{k}{c}(y - m_* x)\right), \quad \Phi\left(\frac{k}{c}(y + m_* x)\right)$$

$$m_* = \frac{\sqrt{c^2 - k^2}}{k} > 0$$



$$v_0(x, y) = \max\left\{ \Phi\left(\frac{k}{c}(y - m_* x)\right), \Phi\left(\frac{k}{c}(y + m_* x)\right) \right\}$$

$$= \Phi\left(\frac{k}{c}(y - m_* |x|)\right)$$

Theorem (Two-dimensional V-form front)

There exists a unique V such that one has

$$\Delta V + c\frac{\partial V}{\partial y} + f(V) = 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$\lim_{R\to\infty} \sup_{x^2+y^2 \geq R^2} |V(x, y) - v_0(x, y)| = 0$$

$$0 < V(x, y) < 1, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

We call V the two-dimensional V-form front associated with $\{ y = m_* |x| \}$.

Supersolutions and subsolutions

supersolution $\bar{v}$

$$L[\bar{v}] = -\Delta \bar{v} - c\frac{\partial \bar{v}}{\partial y} - f(\bar{v}) \geqq 0,$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

subsolution $\underline{v}$

$$L[\underline{v}] = -\Delta \underline{v} - c\frac{\partial \underline{v}}{\partial y} - f(\underline{v}) \leqq 0,$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

weak supersolution $\bar{v}(x, y)$

$$w(x, y, t; \bar{v}) \leqq \bar{v}(x, y), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \ t > 0$$

weak subsolution $\underline{v}(x, y)$

$$\underline{v}(x, y) \leqq w(x, y, t; \underline{v}), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \ t > 0$$

A supersolution is a weak supersolution.
(sub) (sub)

$v_0(x, y)$ is a weak subsolution

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Assume a weak supersolution $\bar{v}$ and

a weak subsolution $\underline{v}$ satisfy

$$\underline{v}(x, y) \leqq \bar{v}(x, y), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in R^2$$

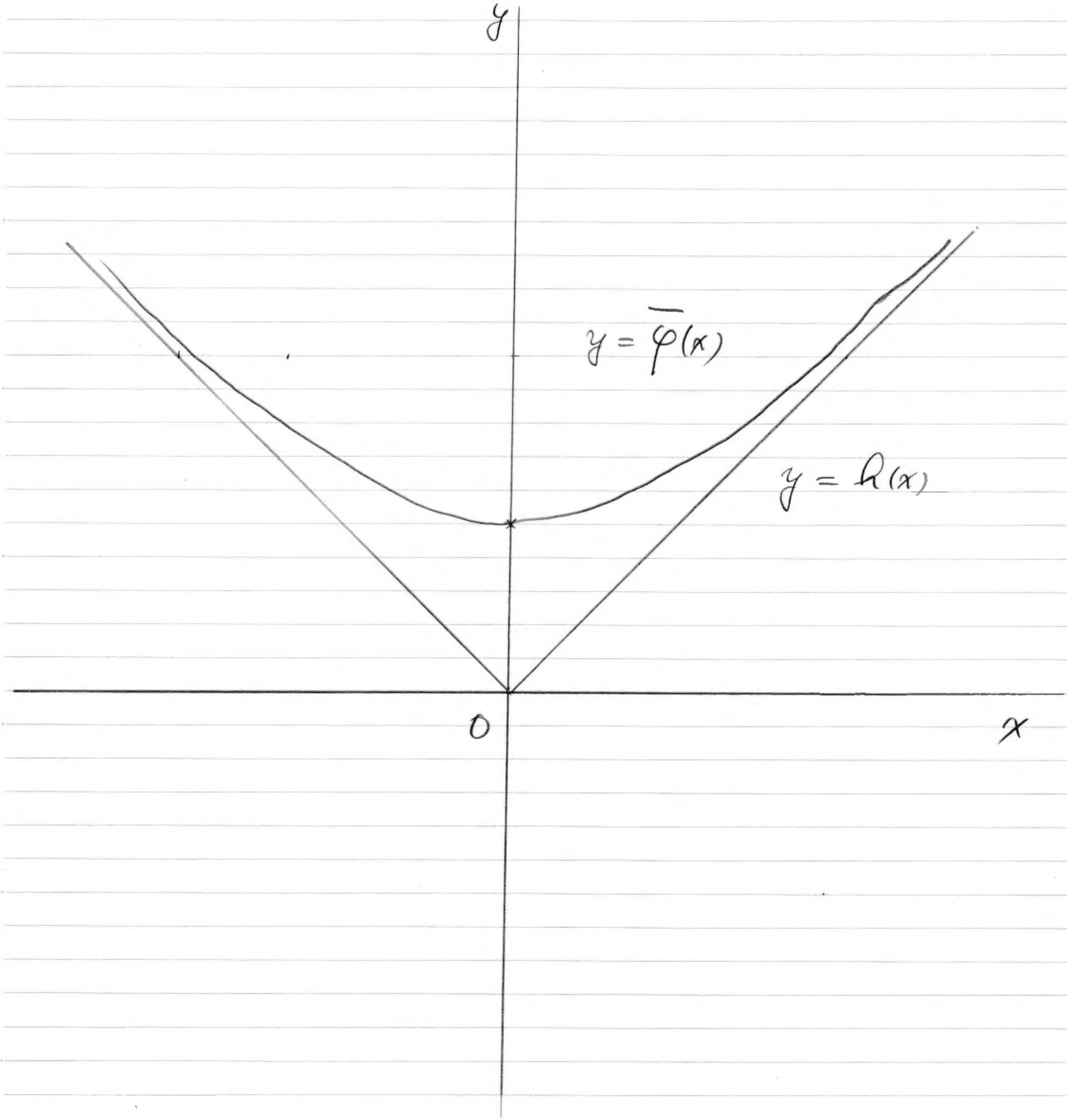
Them there exists a solution V with

$$L[V] = \Delta V + c \frac{\partial V}{\partial y} + f(V) = 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$\underline{v}(x, y) \leqq V(x, y) \leqq \bar{v}(x, y), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

(Sattinger)

$h(x) = m_* |x|, \quad x \in \mathbb{R}$



Construction of $\varphi(x)$

mollifier $\rho(x) = \dfrac{\operatorname{sech} x}{\pi}$

$\bar{\varphi} = \rho * h$ (convolution)

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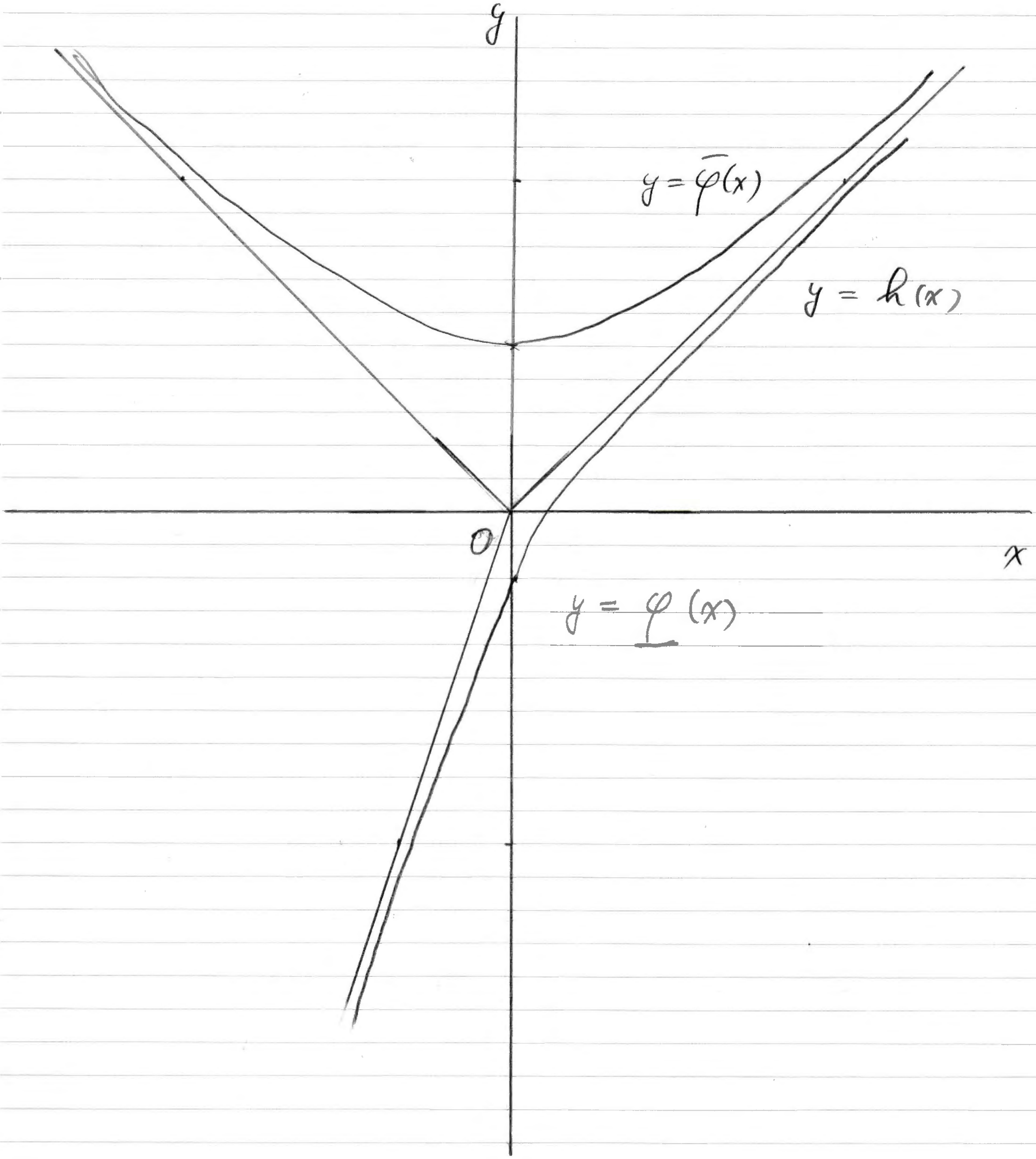


Construction of $\underline{\varphi}(x)$

$$\underline{h}(x) = \begin{cases} m_* x & \text{if } x \geq 0 \\ 2m_* x & \text{if } x < 0 \end{cases}$$

$$\underline{\varphi} = \rho * \underline{h} \quad (\text{convolution})$$

$$\beta = \frac{1}{2}\min\left\{-f'(0),\ -f'(1)\right\} > 0$$

Take $\delta_* \in (0, \frac{1}{8})$ with

$$-f'(s) > \beta \quad \text{if} \quad \min\left\{|s|, |s-1|\right\} < 2\delta_*$$

$$M = \max_{-\delta_* \leqq s \leqq 1+\delta_*} |f'(s)| > 0$$

$$m_1 = \min_{\delta_* \leqq \Phi(\mu) \leqq 1-\delta_*} \left(-\Phi'(\mu)\right) > 0$$

Take $K_0 > 0$, $\kappa_0 > 0$ with

$$\max\left\{|\Phi'(\mu)|, |\mu\Phi'(\mu)|, |\Phi''(\mu)|\right\} \leqq K_0 e^{-\kappa_0|\mu|}$$

$$\mu \in \mathbb{R}$$

Take $\rho_* > 0$ with

$$\beta m_1 \rho_* > \beta + M$$

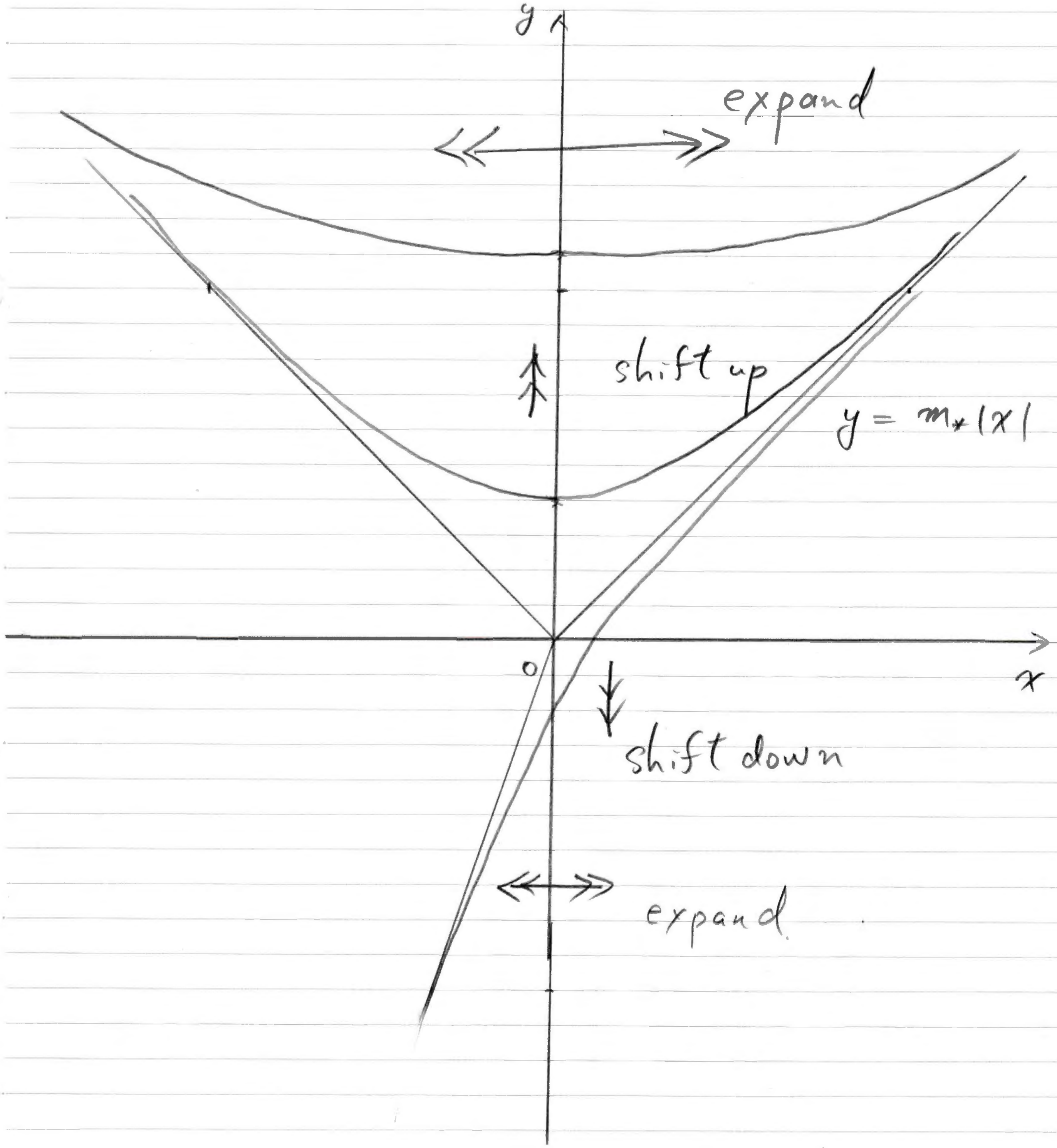
Let ε be arbitrarily given with

$$0 < \varepsilon < \min\left\{ \frac{1}{2}, \frac{\delta_*}{2C}, \frac{K_0}{k}, \frac{m_1}{4M} \right\}$$

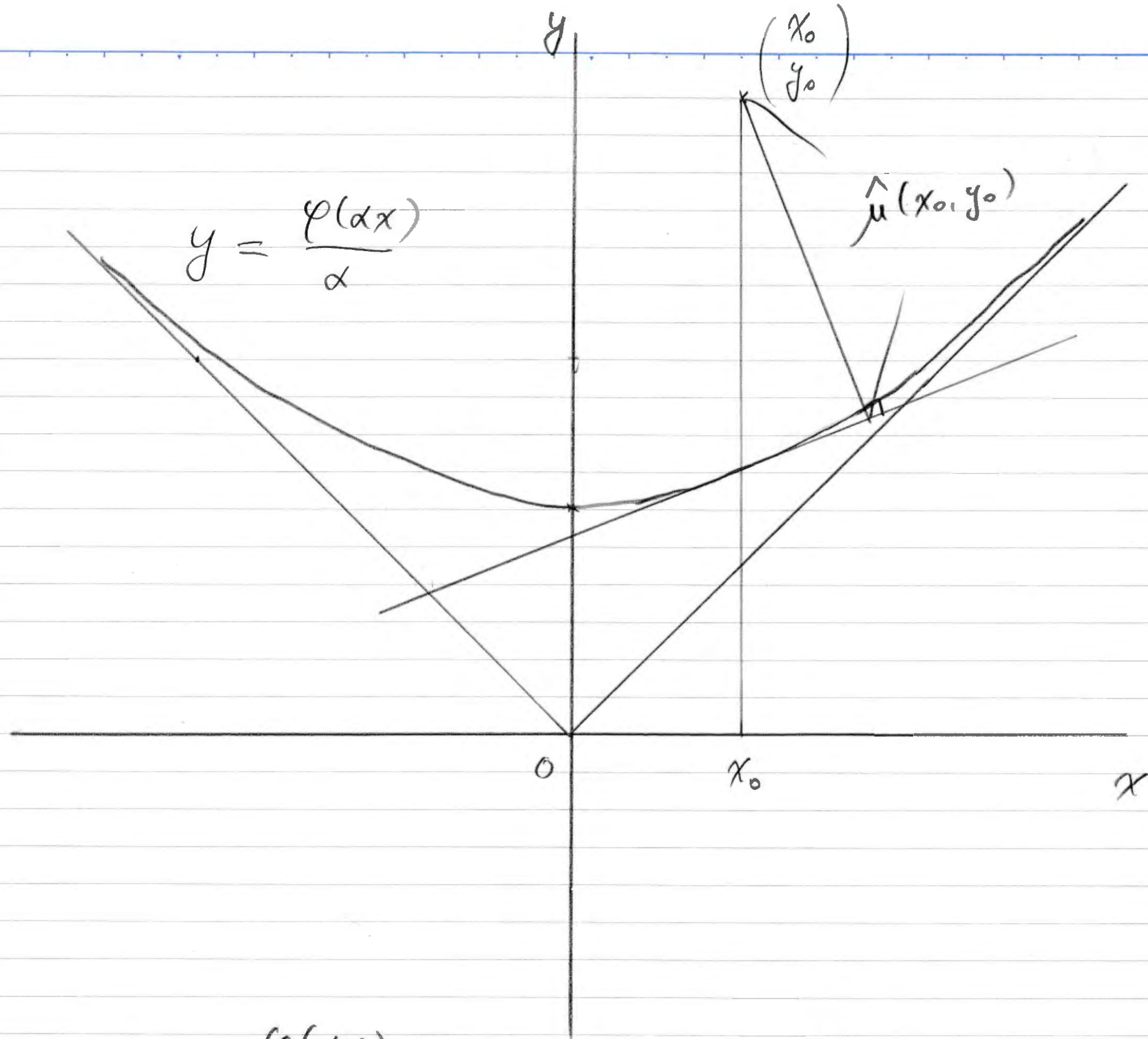
Let φ be either $\overline{\varphi}$ or $\underline{\varphi}$ hereafter.

$\alpha \in (0,1)$

Transformation $\varphi(x) \longmapsto \dfrac{\varphi(\alpha x)}{\alpha}$



$k(x) = m_* |x|$ is invariant under this transformation.



$$y - \frac{\varphi(\alpha x_0)}{\alpha} = \varphi'(\alpha x_0)(x - x_0)$$

$$\hat{\mu}(x_0, y_0; \varphi) = \frac{y_0 - \frac{\varphi(\alpha x_0)}{\alpha}}{\sqrt{1 + \varphi'(\alpha x_0)^2}}$$

$$\overline{S}(x) = \frac{c}{\sqrt{1+\overline{\varphi}'(x)^2}} - k > 0, \quad x \in \mathbb{R}$$

$$\underline{S}(x) = \frac{c}{\sqrt{1+\underline{\varphi}'(x)^2}} - k < 0, \quad x \in \mathbb{R}$$

<u>Construction of a super (sub) solution</u>

$$\overline{v}(x,y) = \Phi(\hat{\mu}(x,y;\overline{\varphi})) + \varepsilon \overline{S}(\alpha x)$$

$$\underline{v}(x,y) = \Phi(\hat{\mu}(x,y;\underline{\varphi})) + \varepsilon \underline{S}(\alpha x)$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \alpha \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\overline{v}(x,y) = \Phi\left(\frac{\eta - \overline{\varphi}(\xi)}{\alpha\sqrt{1+\overline{\varphi}'(\xi)^2}} \right) + \varepsilon \overline{S}(\xi)$$

$$\underline{v}(x,y) = \Phi\left(\frac{\eta - \underline{\varphi}(\xi)}{\alpha\sqrt{1+\underline{\varphi}'(\xi)^2}} \right) + \varepsilon \underline{S}(\xi)$$

supersolution and subsolution written by rescaled coordinate (ξ, η).

Hereafter let (φ, S) be either

$(\bar{\varphi}, \bar{S})$ or $(\underline{\varphi}, \underline{S})$.

$$L[v] = -\Delta v - c\frac{\partial v}{\partial y} - f(v).$$

We calculate $L[\bar{v}]$ and $L[\underline{v}]$.

Hereafter let v be either $\bar{v}$ or $\underline{v}$.

$$L[v] = -\Phi''(\hat{\mu}(x,y;\varphi)) - \frac{c}{\sqrt{1+\varphi'(\xi)^2}}\Phi'(\hat{\mu}(x,y;\varphi))$$

$$- f(v) + \alpha R(\xi, \eta, \hat{\mu}(x,y;\varphi), \varepsilon, \alpha)$$

$$R(\xi, \eta, \mu, \varepsilon, \alpha; \varphi) = -G(\xi)\Phi'(\mu) - \alpha H(\xi)\mu\Phi'(\mu)$$

$$+ \frac{2\varphi'(\xi)F(\xi)}{\sqrt{1+\varphi'(\xi)^2}}\mu\Phi''(\mu) - \alpha F(\xi)^2\Phi''(\mu) - \alpha\varepsilon\frac{\partial^2 S}{\partial \xi^2}(\xi)$$

$$F(\xi) = \sqrt{1+\varphi'(\xi)^2}\,\frac{\partial}{\partial \xi}\left(\frac{1}{\sqrt{1+\varphi'(\xi)^2}}\right)$$

$$G(\xi) = -\frac{\partial}{\partial \xi}\left(\frac{\varphi'(\xi)}{\sqrt{1+\varphi'(\xi)^2}}\right) - \frac{\varphi'(\xi)}{\sqrt{1+\varphi'(\xi)^2}}F(\xi)$$

$$H(\xi) = \frac{\partial F}{\partial \xi}(\xi) + F(\xi)^2$$

A

$$= \max \left\{ \sup_{(\xi,\eta,\mu,\varepsilon,\alpha)\in\mathbb{R}^3\times(0,1)^2} \frac{|R(\xi,\eta,\mu,\varepsilon,\alpha;\overline{\varphi})|}{\overline{S}(\xi)}, \sup_{(\xi,\eta,\mu,\varepsilon,\alpha)\in\mathbb{R}^3\times(0,1)^2} \frac{|R(\xi,\eta,\mu,\varepsilon,\alpha;\underline{\varphi})|}{-\underline{S}(\xi)} \right\}$$

One has

$$\underline{0 \leq A < \infty}$$

Recalling

$$S(\xi) = \frac{c}{\sqrt{1+\varphi'(\xi)^2}} - k,$$

we have

$$-\Phi''(\hat{\mu}) - \frac{c}{\sqrt{1+\varphi'(\xi)^2}}\Phi'(\hat{\mu}) - f(v)$$

$$= \left(-\Phi'(\hat{\mu}) - \varepsilon\int_0^1 f'(\Phi(\hat{\mu}) + \theta\varepsilon S(\xi))\,d\theta\right) S(\xi)$$

$$\frac{L[v]}{\delta(\xi)} = -\Phi'(\hat{\mu}) - \varepsilon\int_0^1 f'(\Phi(\hat{\mu}) + \theta\varepsilon\delta(\xi))d\theta$$

$$+ \alpha \frac{R(\xi, \eta, \hat{\mu}; \varepsilon, \alpha; \varphi)}{\delta(\xi)}$$

$$\geqq -\Phi'(\hat{\mu}) - \varepsilon\int_0^1 f'(\Phi(\hat{\mu}) + \theta\varepsilon\delta(\xi))d\theta$$

$$-\alpha A$$

$$> 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Thus we obtain

$$L[\bar{v}] > 0, \quad L[\underline{v}] < 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

<u>Proposition 1</u>

$\bar{v}$ is a supersolution,

$\underline{v}$ is a subsolution.

$$0 < \alpha < \min \left\{ \frac{1}{2},\ \frac{\varepsilon\beta}{2A},\ \frac{m_1}{4A},\ \frac{K_0 k^2 \bar{\nu}}{c \log\left(\frac{K_0}{k\varepsilon}\right)},\ \frac{c K_0 \nu}{\left(1+4(m_*)^2\right)\log\left(\frac{K_0\sqrt{1+4(m_*)^2}}{c\varepsilon}\right)} \right\}$$

ν, $\bar{\nu}$ are positive constants

$$m_1 = \min_{\delta_* \leqq \Phi(x) \leqq 1-\delta_*} \left(-\Phi'(x)\right) > 0.$$

Proposition 2

$$\underline{v}(x,y) < v_0(x,y) < \bar{v}(x,y), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Proof of $\underline{v} < v_0$

It suffices to prove

$$\Phi\left(\frac{k}{c}(y - m_* x)\right) \leqq \bar{v}(x,y)$$

If

$$\hat{\mu} \leqq \frac{k}{c}(y - m_* x),$$

we get

$$\Phi\left(\frac{k}{c}(y - m_* x)\right) \leqq \Phi(\hat{\mu}) < \bar{v}(x,y).$$

Thus it suffices to assume

$$\hat{\mu} > \frac{k}{c}(y - m_* x).$$

Now

$$\hat{\mu} = \frac{y - \frac{\varphi(\alpha x)}{\alpha}}{\sqrt{1 + \varphi'(\alpha x)^2}}$$

Then

$$\frac{y - m_* x + m_* x - \frac{\varphi(\alpha x)}{\alpha}}{\sqrt{1+\varphi'(\xi)^2}} > \frac{k}{c}(y - m_* x).$$

that is

$$\delta(\xi)(y - m_* x) > \frac{c}{\alpha\sqrt{1+\varphi'(\xi)^2}}\left(\varphi(\xi) - m_* \xi\right)$$

$$y - m_* x > \frac{c}{\alpha\sqrt{1+\varphi'(\xi)^2}} \frac{\varphi(\xi) - m_* \xi}{\delta(\xi)} \geq \frac{c\nu_1}{\alpha\sqrt{1+\varphi'(\xi)^2}}$$

Here $\nu_1 \in (0, \infty)$ is a constant.

Thus we have

$$y - m_* x > \frac{k\nu_1}{\alpha}.$$

Now we have

$$\hat{\mu} = \frac{y - \frac{\varphi(\alpha x)}{\alpha}}{\sqrt{1+\varphi'(\xi)^2}} \leq \frac{y - m_* x}{\sqrt{1+\varphi'(\xi)^2}}$$

$$\Phi\left(\frac{y-m_* x}{\sqrt{1+\varphi'(\xi)^2}}\right) \leqq \Phi(\hat{\mu})$$

Then we have

$$\begin{aligned}
&\overline{V}(x,y) - V_0(x,y) \\
&\geqq \Phi\left(\frac{y-m_* x}{\sqrt{1+\varphi'(\xi)^2}}\right) + \varepsilon S(\xi) - \Phi\left(\frac{k}{c}(y-m_* x)\right) \\
&\geqq S(\xi)\left(\varepsilon - \frac{1}{c}\sup_{\mu \geq \frac{k\nu_1}{\alpha}} \left|\mu \Phi'\left(\frac{k}{c}\mu\right)\right|\right)
\end{aligned}$$

Thus we obtain

$$\overline{V}(x,y) - V_0(x,y) \geqq \frac{1}{2}\varepsilon S(\xi) > 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Proof of $\underline{v}(x,y) < v_0(x,y)$

It suffices to prove

$$\underline{v}(x,y) < \Phi\left(\frac{k}{c}(y - m_* x)\right).$$

If

$$\hat{\mu} \geqq \frac{k}{c}(y - m_* x),$$

we have

$$\Phi(\hat{\mu}) \leqq \Phi\left(\frac{k}{c}(y - m_* x)\right)$$

$$\underline{v}(x,y) = \Phi(\hat{\mu}) + \varepsilon \underline{S}(\xi) < \Phi\left(\frac{k}{c}(y - m_* x)\right)$$

$$\ominus$$

Thus it suffices to assume

$$\hat{\mu} < \frac{k}{c}(y - m_* x).$$

$$\hat{\mu} = \frac{y - \dfrac{\varphi(\alpha x)}{\alpha}}{\sqrt{1 + \varphi'(\xi)^2}}$$

We write $\underline{\varphi}$ as φ here.

$$\frac{y - m_* x + m_* x - \frac{\varphi(\alpha x)}{\alpha}}{\sqrt{1+\varphi'(\xi)^2}} < \frac{h}{c}(y - m_* x)$$

$$\frac{c}{\alpha\sqrt{1+\varphi'(\xi)^2}}\underbrace{\left(m_* \xi - \varphi(\xi)\right)}_{\oplus} < \underbrace{\left(-\underline{S}(\xi)\right)}_{\oplus}(y - m_* x)$$

$$0 < \underline{\nu} \leq \frac{m_* x - \varphi(x)}{-\underline{S}(x)}$$

$$1 + (m_*)^2 < 1 + \varphi'(\xi)^2 < 1 + 4(m_*)^2$$

$$\frac{c\,\underline{\nu}}{\alpha\sqrt{1 + 4(m_*)^2}} < y - m_* x$$

Now we have

$$\varphi(x) \leq m_* x$$

$$\hat{\mu} = \frac{y - \frac{\varphi(\alpha x)}{\alpha}}{\sqrt{1+\varphi'(\xi)^2}} \leq \frac{y - m_* x}{\sqrt{1+\varphi'(\xi)^2}}$$

Thus we find

$$\Phi(\hat{\mu}) \leqq \Phi\left(\frac{y - m_* x}{\sqrt{1+\varphi'(\xi)^2}}\right).$$

Then we obtain

$$\Phi\left(\frac{k}{c}(y - m_* x)\right) - \underline{\nu}(x, y)$$

$$= \Phi\left(\frac{k}{c}(y - m_* x)\right) - \Phi(\hat{\mu}) - \varepsilon\, \underline{S}(\xi)$$

$$\geqq \Phi\left(\frac{k}{c}(y - m_* x)\right) - \Phi\left(\frac{y - m_* x}{\sqrt{1+\varphi'(\xi)^2}}\right) - \varepsilon\, \underline{S}(\xi)$$

$$= \left(-\underline{S}(\xi)\right) \times$$

$$\left(\varepsilon - \frac{y - m_* x}{c}\int_0^1 \Phi'\left(\left(\frac{k}{c}\theta + \frac{(1-\theta)}{\sqrt{1+\varphi'(\xi)^2}}\right)(y - m_* x)\right) d\theta\right)$$

Using

$$y - m_* x > \frac{c\,\underline{\nu}}{\alpha\sqrt{1+\varphi(m_*)^2}}$$

we obtain

$$\Phi\left(\frac{k}{c}(y - m_* x)\right) - \underline{\nu}(x, y) > 0.$$

Theorem (V-form traveling front)

For every $c \in (k, \infty)$, there exists $V(x, y)$ such that one has

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + c\frac{\partial V}{\partial y} + f(V) = 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

$$\lim_{R \to \infty} \sup_{x^2+y^2 \geqq R^2} \left| V(x, y) - \Phi\left(\frac{k}{c}\left(y - \frac{\sqrt{c^2-k^2}}{k}|x|\right)\right)\right| = 0$$

$$0 < V(x, y) < 1, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Under the conditions stated above, V is uniquely determined.

$$\mathcal{L}[w] = w_t - w_{xx} - w_{yy} - c w_y - f(w)$$

Lemma $\forall \delta \in (0, \delta_*)$

$\overline{w}(x,y,t)$
$= V(x, y - \rho_* \delta (1 - e^{-\beta t})) + \delta e^{-\beta t}$ is a supersolution, that is,

$$\mathcal{L}[\overline{w}] \geq 0 \quad \text{for} \quad \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,\ t > 0.$$

Proof We write $Y = y - \rho_* \delta (1 - e^{-\beta t})$

$$\mathcal{L}[\overline{w}] = -\rho_* \delta \beta e^{-\beta t} V_y(x, Y) - \delta \beta e^{-\beta t}$$
$$- V_{xx}(x, Y) - V_{yy}(x, Y) - c V_y(x, Y) - f(V(x, Y) + \delta e^{-\beta t})$$

$$= -\rho_* \delta \beta e^{-\beta t} V_y(x, Y) - \delta \beta e^{-\beta t}$$
$$+ f(V(x, Y)) - f(V(x, Y) + \delta e^{-\beta t})$$

Now
$$\frac{d}{d\theta}\left(f(V + \delta \theta e^{-\beta t}) \right) = f'(V + \delta \theta e^{-\beta t}) \delta e^{-\beta t}$$

$$f(V + \delta e^{-\beta t}) - f(V) = \delta e^{-\beta t} \int_0^1 f'(V + \delta \theta e^{-\beta t})\, d\theta.$$

Thus

$$\mathcal{L}[\overline{w}] = \delta e^{-\beta t} \left(-\rho_* \beta V_y(x, Y) - \int_0^1 f'(V(x, Y) + \delta \theta e^{-\beta t})\, d\theta - \beta \right)$$

$$|\delta \theta e^{-\beta t}| < \delta_*$$

If $V(x, Y) \leqq -1 + \delta_*$ or $1 - \delta_* \leqq V(x, Y)$, we have

$$-f'(V(x, Y) + \delta\theta e^{-\beta t}) > \beta$$

$$\mathcal{L}[\bar{w}] \geqq \delta e^{-\beta t}\left(-\rho_* \beta V_y(x, Y) + \beta - \beta\right) > 0$$

If $-1 + \delta_* < V(x, Y) < 1 - \delta_*$, we have

$$-\rho_* \beta V_y(x, Y) - \int_0^1 f'(V(x, Y) + \delta\theta e^{-\beta t})\, d\theta - \beta$$

$$\geqq \rho_* \beta \min_{-1+\delta_* \leqq \Phi(\mu) \leqq 1-\delta_*} (-\Phi'(\mu)) - \max_{|s| \leqq 2} |f'(s)| - \beta > 0$$

Thus we obtain

$$\mathcal{L}[\bar{w}] > 0 \quad \text{in } \mathbb{R}^2. \quad //$$

$$\eta = y - \sigma\delta(1 - e^{-\beta t})$$

$$-\sigma\beta V_y(x,\eta) - \int_0^1 f'(\bar{V}(x,\eta) + \delta\theta e^{-\beta t})\,d\theta - \beta$$

$$\geq \sigma\beta \min_{-1+\delta_* \leq V(x,\eta) \leq 1-\delta_*} (-V_y(x,\eta)) - M - \beta > 0$$

$$\min_{-1+\delta_* \leq V(x,y) \leq 1-\delta_*} (-V_y(x,y)) > \gamma > 0$$

$$\sigma\beta \min_{-1+\delta_* \leq V(x,y) \leq 1-\delta_*} (-V_y(x,y)) > M + \beta$$

Proof of the theorem

It suffices to prove $V \equiv U$. We have

$$\Phi\left(\frac{k}{c}(y - m_*(|x|))\right) < V(x,y) \leqq U(x,y) \leqq$$

$$\min\left\{1,\ \Phi\left(\frac{y - \frac{\varphi(\alpha x)}{\alpha}}{\sqrt{1+\varphi'(\alpha x)^2}}\right) + \varepsilon\left(\frac{c}{\sqrt{1+\varphi'(\alpha x)^2}} - k\right)\right\}$$

This gives

$$\lim_{R\to\infty} \sup_{x^2+y^2 \geqq R^2} |U(x,y) - V(x,y)| = 0.$$

Then if $\lambda > 0$ is large enough we have

$$U(x,y) \leqq V(x, y-\lambda) + \delta_* \quad \text{in } \mathbb{R}^2$$

$$U(x,y) \leqq V(x, y - \lambda - \rho_* \delta_* (1 - e^{-\beta t})) + \delta_* e^{-\beta t} \Big|_{t=0}$$

$$w(x,y,t;U) \leqq V(x, y - \lambda - \rho_* \delta_* (1 - e^{-\beta t})) + \delta_* e^{-\beta t}$$

$$\overset{\|}{U(x,y)}$$

Sending $t \to +\infty$, we have

$$U(x,y) \leqq V(x, y - \lambda - \rho_* \delta_*)$$

Define $\Lambda \geqq 0$ by

$$\Lambda = \inf\left\{\lambda \geqq 0 \ \middle|\ U(x,y) \leqq V(x, y-\lambda) \text{ for } {}^\forall \binom{x}{y} \in \mathbb{R}^2\right\}$$

If $\Lambda = 0$, we get $V \equiv U$. Assume $\Lambda > 0$ and get a contradiction

$$V(x,y) \underset{\not\equiv}{\leq} U(x,y) \underset{\not\equiv}{\leq} V(x, y-\Lambda)$$

We take $R_0 > 0$ large enough such that

$$\sup_{|y-h(x)| \geq R_0 - \Lambda - 1} |V_y(x,y)| < \frac{1}{2p_*}$$

For any $h \in (0, \frac{1}{2p_*})$ we get

$$V(x, y-\Lambda+2p_* h) - V(x, y-\Lambda)$$
$$= 2p_* h \int_0^1 V_y(x, y-\Lambda+2\theta p_* h)\, d\theta > -h \quad \cdots (1)$$

if $|y - h(x)| \geq R_0$.

We have

$$\lim_{R\to\infty} \sup_{x^2+y^2 \geq R^2} \left| U(x,y) - \Phi\left(\frac{k}{c}(y - m_*|x|)\right) \right| = 0$$

$$\lim_{R\to\infty} \sup_{x^2+y^2 \geq R^2} \left| V(x, y-\Lambda+2p_* h) - \Phi\left(\frac{k}{c}(y - m_*|x| - \Lambda + 2p_* h)\right) \right| = 0.$$

We take $h \in (0, \min\{\frac{1}{2p_*}, \delta_*\})$ small enough such that

$$U(x,y) < V(x, y-\Lambda+2p_* h) \quad \text{if} \quad |y - h(x)| \leq R_0$$
$$\cdots (2)$$

Combining ① and ②, we get

$$U(x,y) \leqq V(x, y-\Lambda+2\rho_* h)+h$$

$$w(x,y,t;U) \leqq V(x, y-\Lambda+2\rho_* h-\rho_* h(1-e^{-\beta t}))+he^{-\beta t}$$

Letting $t \to +\infty$, we have

$$U(x,y) \leqq V(x, y-\Lambda+\rho_* h).$$

This contradicts the definition of Λ. //

Pyramidal traveling fronts

$f \in C^1[0,1]$

$f(0)=0, \quad f(1)=0$

$f'(0)<0, \quad f'(1)<0$

$$\int_0^1 f(s)\,ds > 0$$

Assume that there exists (k, Φ) with

$$\begin{cases} \Phi''(\mu) + k\Phi'(\mu) + f(\Phi(\mu)) = 0, & \mu \in \mathbb{R} \\ \Phi(-\infty) = 1, \quad \Phi(+\infty) = 0 \end{cases}$$

Example

$f(u) = -u(u+a)(u-1), \quad 0 < a < \frac{1}{2}$

$k = \dfrac{1-2a}{\sqrt{2}}, \quad \Phi(\mu) = \dfrac{1}{2} - \dfrac{1}{2}\tanh\left(\dfrac{\mu}{2\sqrt{2}}\right)$

One has $\Phi'(\mu) < 0, \quad \mu \in \mathbb{R}$

(Fife–McLeod 1977)

$n \geq 1$

reaction–diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tilde{\Delta} u + f(u), & \tilde{x} \in \mathbb{R}^{n+1}, \quad t > 0 \\ u(x, 0) = u_0(x), & \tilde{x} \in \mathbb{R}^{n+1} \end{cases}$$

$$\tilde{\Delta} = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2}$$

$u_0 \in BU(\mathbb{R}^{n+1})$

We write the solution as $u(\tilde{x}, t; u_0)$.

$$\tilde{x} = \begin{pmatrix} x \\ \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}, \quad t > 0$$

$$\tilde{\Delta} = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2}, \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

$c \in (k, \infty)$ arbitrarily given

$$m_* = \frac{\sqrt{c^2 - k^2}}{k} \in (0, \infty)$$

$m \geq 2$

Let

$\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_m \in \mathbb{R}^n$

be avitrarily given with

$\mathbf{a}_i \neq \mathbf{a}_j$ if $i \neq j$.

$$h_j(\mathbf{x}) = m_* (\mathbf{a}_j, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

$$h(\mathbf{x}) = \max_{1 \leq j \leq m} h_j(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

$\{ (\mathbf{x}, x_{n+1}) \mid x_{n+1} \geq h(\mathbf{x}) \}$ is a pyramid in $\mathbb{R}^{n+1}$!

Example $n = 3$

$$h(x_1, x_2) = \frac{m_*}{\sqrt{2}} \left(|x_1| + |x_2| \right)$$

$\{ x_3 \geqq h(x_1, x_2) \}$ is a square pyramid in $\mathbb{R}^3$.

$$z = x_{n+1} - ct$$

$$u(x, x_{n+1}, t) = w(x, z, t)$$

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w - \dfrac{\partial^2 w}{\partial z^2} - c\dfrac{\partial w}{\partial z} - f(w) = 0, \\ \qquad\qquad\qquad \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{n+1},\ t > 0 \\ w(x, z, 0) = u_0(x, z), \quad \begin{pmatrix} x \\ z \end{pmatrix} \in \mathbb{R}^{n+1} \end{cases}$$

We write z simply as x_{n+1}.

$$\begin{cases} \dfrac{\partial w}{\partial t} - \tilde{\Delta} w - c\dfrac{\partial w}{\partial x_{n+1}} - f(w) = 0, \\ \qquad\qquad\qquad \tilde{x} \in \mathbb{R}^{n+1},\ t > 0 \\ w(\tilde{x}, 0) = u_0(\tilde{x}), \quad \tilde{x} \in \mathbb{R}^{n+1} \end{cases}$$

We write the solution as $w(\tilde{x}, t; u_0)$.

Profile equation

$$\tilde{\Delta} v + c \frac{\partial v}{\partial x_{n+1}} + f(v) = 0, \quad \tilde{x} \in \mathbb{R}^{n+1}$$

Planar traveling front

$$\Phi\left(\frac{x_{n+1} - h_j(x)}{\sqrt{1 + (m_*)^2}} \right)$$

weak subsolution

$$v_0(x, x_{n+1}) = \max_{1 \leq j \leq m} \Phi\left(\frac{x_{n+1} - h_j(x)}{\sqrt{1 + (m_*)^2}} \right)$$

$$= \Phi\left(\frac{k}{c} \left(x_{n+1} - h(x) \right) \right)$$

Theorem (pyramidal traveling front)

There exists $V(x, x_{n+1})$ with

$$\tilde{\Delta} V + c\frac{\partial V}{\partial x_{n+1}} + f(V) = 0, \quad \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$$

$$\lim_{R\to\infty} \sup_{|\tilde{x}| \geq R} \left| V(x, x_{n+1}) - \Phi\left(\frac{k}{c}(x_{n+1} - h(x))\right)\right| = 0$$

$$0 < V(x, x_{n+1}) < 1, \quad \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$$

Moreover, V is uniquely determined under conditions stated above.

We call V the pyramidal traveling front associated with a pyramid $\{x_{n+1} \geq h(x)\}$.

mollified pyramid $x_{n+1} = \varphi(x)$

method 1

$$\tilde{\rho} \in C^{\infty}[0, \infty)$$

(1) $\tilde{\rho}(r) > 0, \quad \tilde{\rho}_r(r) \leqq 0, \quad r \geqq 0$

(2) $\tilde{\rho}(r) \equiv 1$ if $r > 0$ is small enough

(3) $\tilde{\rho}(r) = e^{-r}$ if $r > 0$ is large enough

(4) $\displaystyle\int_{\mathbb{R}^n} \tilde{\rho}(|x|)\, dx = 1$

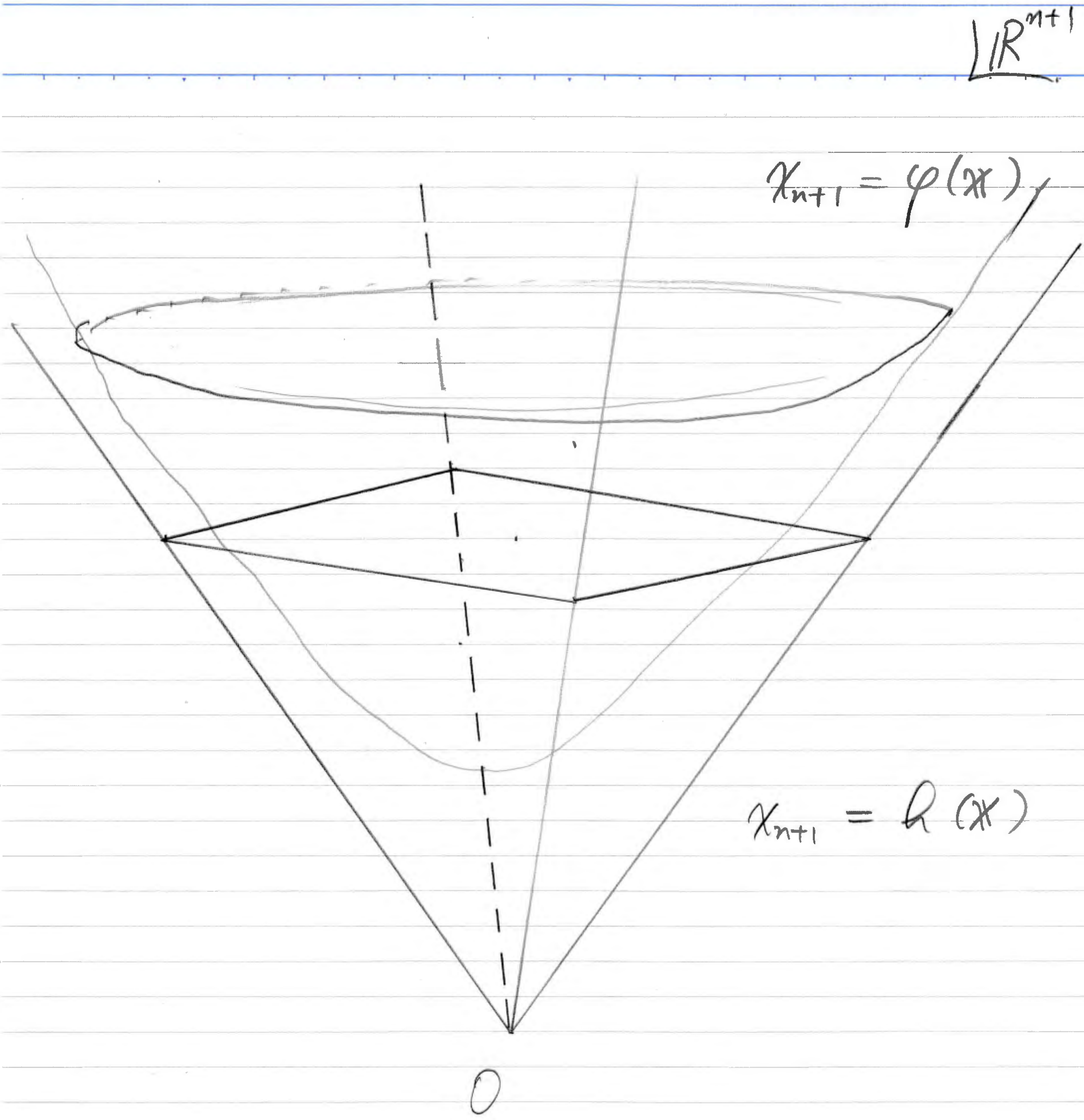
We define $\rho(x) = \tilde{\rho}(|x|), \quad x \in \mathbb{R}^n$

$$\varphi = \rho * h$$

method 2

$$\varphi(x) = \log\left(\sum_{j=1}^{m} \exp(h_j(x))\right),$$

$$x \in \mathbb{R}^n$$



$$S(x) = \frac{c}{\sqrt{1+|\nabla\varphi(x)|^2}} - k, \quad x \in \mathbb{R}^n$$

$\alpha \in (0, 1)$

$$\tilde{\boldsymbol{\xi}} = \alpha \tilde{\boldsymbol{x}}, \quad \tilde{\boldsymbol{x}} \in \mathbb{R}^{n+1}$$

that is

$$\begin{pmatrix} \boldsymbol{\xi} \\ \xi_{n+1} \end{pmatrix} = \alpha \begin{pmatrix} \boldsymbol{x} \\ x_{n+1} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{x} \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$$

Consider

$$x_{n+1} = \frac{\varphi(\alpha \boldsymbol{x})}{\alpha}, \quad \boldsymbol{x} \in \mathbb{R}^n$$

that is

$$\xi_{n+1} = \varphi(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{R}^n$$

$\mathbb{b} \in \mathbb{R}^n$

the tangential plane of

$$x_{n+1} = \frac{\varphi(\alpha \mathbb{x})}{\alpha}$$

at $\left(\mathbb{b}, \frac{\varphi(\alpha \mathbb{b})}{\alpha}\right)$ is given by

$$-(\nabla\varphi(\alpha \mathbb{b}), \mathbb{x} - \mathbb{b}) + x_{n+1} - \frac{\varphi(\alpha \mathbb{b})}{\alpha} = 0.$$

The signed length of the perpendicular to this plane from $(\mathbb{b}, b_{n+1})$ is

$$\frac{b_{n+1} - \frac{\varphi(\alpha \mathbb{b})}{\alpha}}{\sqrt{1 + |\nabla\varphi(\alpha \mathbb{b})|^2}}.$$

$$\hat{u}(x, x_{n+1}) = \frac{x_{n+1} - \dfrac{\varphi(\alpha x)}{\alpha}}{\sqrt{1 + |\nabla \varphi(\alpha x)|^2}}$$

$$= \frac{\xi_{n+1} - \varphi(\xi)}{\alpha \sqrt{1 + |\nabla \varphi(\xi)|^2}}$$

Super solution

$$\bar{v}(x, x_{n+1}) = \Phi(\hat{u}(x), x_{n+1}) + \varepsilon S(\alpha x)$$

$$L[v] = -\tilde{\Delta} v - c\frac{\partial v}{\partial x_{n+1}} - f(v)$$

Proposition 1

$\bar{v}$ is a supersolution, that is,

$$L[\bar{v}] > 0 \quad \text{in } \mathbb{R}^{n+1}$$

Proposition 2

$$v_0(\mathbb{x}, x_{n+1}) < \bar{v}(\mathbb{x}, x_{n+1}),$$

$$\begin{pmatrix} \mathbb{x} \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$$

By Sattinger's theorem, there exists V with $L[V] = 0$

$$v_0(\boldsymbol{x}, x_{n+1}) < V(\boldsymbol{x}, x_{n+1}) < \bar{v}(\boldsymbol{x}, x_{n+1})$$

$$\begin{pmatrix} \boldsymbol{x} \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$$